Clumsy packings with polyominoes

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For a set D of polyominoes, a packing of the plane with D is a maximal set of copies of polyominoes from D that are not overlapping. A packing with smallest density is called **clumsy** packing. We give an example of a set D such that any clumsy packing is aperiodic. In addition, we compute the smallest possible density of a clumsy packing when D consist of a single polyomino of a given size and show that one could always construct a periodic packing arbitrarily close in density to the clumsy packing.

1 Introduction

Tiling the plane with polyominoes is a fascinating task occupying the minds of mathematicians, computer scientists, chemists and many others who simply enjoy nice puzzles. For the extensive literature on the topic, see [1, 4, 7, 8, 11, 12]. Most of the tilings by polyominoes known until recently are periodic. It is believed, see [10] that only in 1994 Penrose gave an example of a small set of polyominoes tiling the plane aperiodically. The mathematical theory of aperiodic tilings lead to research on quasicrystals see [13], that in turn resulted in a Nobel price in chemistry awarded to Shechtman in 2011. Gyárfás, Lehel and Tuza [9] asked about a different type of polyomino arrangements in the plane. They called a set of disjoint polyominoes a **clumsy packing** if no other polyomino can be added without an overlap and the total density is as small as possible. In that paper connected polyominoes of order 2, i.e., dominoes were considered.

Here, we study clumsy packings of general polyominoes. For a more detailed account of the topic, see Walzer [15]. For a few special polyominoes, see Goddard [6]. Next we provide the formal definitions and the statements of our results.

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A *cell* is a closed unit square in the plane whose sides are aligned with coordinate axes and whose corners have integer coordinates. A *polyomino* is a finite set of cells. The number of cells in the polyomino is its *area* or *order*. If one polyomino D' is obtained from another polyomino D by translation, we say that D' is a *copy* of D. Note that we do not consider rotations here. We shall sometimes refer to a polyomino as a union of its cells, and say that a polyomino intersects a subset of the plane if one of its cells intersects this subset. Let $\mathcal{D}$ be a finite set of polyominoes. A *packing of a subset S (of cells) of the plane with $\mathcal{D}$* is a maximal set of pairwise disjoint copies of a polyomino from $\mathcal{D}$ contained in S. Whenever the set $\mathcal{D}$ of polyominoes is clear from the context, we sometimes call a packing with $\mathcal{D}$ simply a packing. Let B_n be the set of cells that form an $n \times n$ square either centered at the origin (for even n) or centered at a cell with lower left corner at the origin. Note that $|B_n| = n^2$. If $\mathcal{P}$ is a packing of the entire plane, its *density* is

$$\text{density}(\mathcal{P}) = \limsup_{n \to \infty} \frac{\left|\left(\bigcup_{D \in \mathcal{P}} D\right) \cap B_n\right|}{|B_n|}.$$

The density of a packing of a finite area subset S (of cells) of the plane is the fraction of the area occupied by the polyominoes of the packing and the area of S.

For a set of polyominoes $\mathcal{D}$, a packing $\mathcal{P}$ with $\mathcal{D}$ is called **clumsy** if it has the smallest density. We say that a packing $\mathcal{P}$ is *periodic*, if there is a positive integer q such that for any $D \in \mathcal{P}$, $D + (0, q) \in \mathcal{P}$ and $D + (q, 0) \in \mathcal{P}$, where $D + (a, b)$ denotes the translation of D by a vector (a, b). If no such q exists, we say that $\mathcal{P}$ is *aperiodic*. The density of a clumsy packing of the plane with $\mathcal{D}$ is called **clumsiness** of $\mathcal{D}$ and is denoted $\text{clum}(\mathcal{D})$. We prove that the clumsiness is a well-defined notion in a somewhat formal argument, see Appendix. The main results of this paper are the following four theorems.

Theorem 1. *For any set $\mathcal{D}$ of polyominoes and for any $\epsilon > 0$, there is a periodic packing $\mathcal{P}$ with $\mathcal{D}$ such that* $\text{density}(\mathcal{P}) - \text{clum}(\mathcal{D}) \le \epsilon$.

Theorem 2. *There is a set $\mathcal{D}$ of polyominoes such that every clumsy packing with $\mathcal{D}$ is aperiodic.*

The decision problem, see [5], concerning clumsy packings is addressed in the next theorem.

Theorem 3. *The question whether, for a given set $\mathcal{D}$ of polyominoes and a given rational number d, $clum(\mathcal{D}) \le d$, is undecidable.*

A connected polyomino is a polyomino for which the interior of the union of its cells is a connected subset of the plane. Note that a polyomino consisting of two cells that share only a corner is not connected.

Theorem 4. *If $\mathcal{D}$ consists of a single polyomino of order k then $clum(\mathcal{D}) \ge \frac{k}{k^2-k+1} \approx 1/k$. If $\mathcal{D}$ consists of a single connected polyomino of order k then $clum(\mathcal{D}) \ge \frac{k}{k^2-(\lfloor (k-1)/2 \rfloor^2 + \lceil (k-1)/2 \rceil^2)} \approx 2/k$. Both inequalities are tight.*

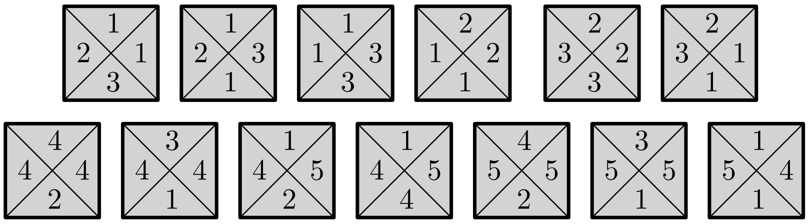

Figure 1: Set of Wang tiles with edges labeled by integers, [2].

One of our tools is sets of Wang tiles. A *Wang tile* is a cell with a label assigned to each of its four sides. See Figure 1 for illustration.

One Wang tile is a *copy* of another Wang tile if one can be obtained from another by translation such that the corresponding labels are preserved. A set $\mathcal{W}$ of Wang tiles *tiles the plane* if the plane is a union of copies of tiles from $\mathcal{W}$ such that any two tiles are either disjoint, share only a corner, or intersect exactly by an edge with the same label. A corresponding set $\mathcal{T}$ of (copies of) Wang tiles is called a *tiling with* $\mathcal{W}$. A tiling $\mathcal{T}$ is *periodic* if there is a positive integer q such that for any tile W in $\mathcal{T}$, $W + (0, q)$ and $W + (q, 0)$ are in $\mathcal{T}$.

Theorem 5 (Culik,[2])**.** *There is a set of* 13 *Wang tiles that tile the plane and that any such tiling is aperiodic. An example of such a set is shown in Figure 1.*

2 Wang tiles constructions and properties and difference sets

Given a set $\mathcal{W}$ of Wang tiles, we shall construct a polyomino with parameter x corresponding to each Wang tile $W \in \mathcal{W}$ as shown in Figure 2. We refer to these polyominoes as $\mathcal{W}$-polyominoes and call the set of all $\mathcal{W}$-polyominoes $\mathcal{D}(\mathcal{W}) = \mathcal{D}(\mathcal{W}, x)$.

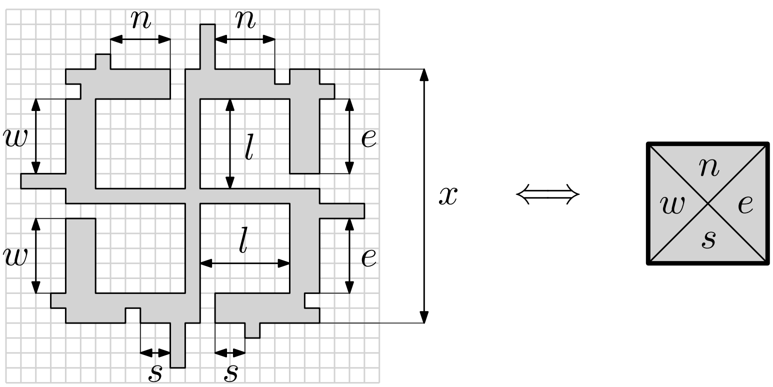

Figure 2: A $\mathcal{W}$-polyomino $D(W)$ corresponding to a Wang tile $W \in \mathcal{W}$ with integral edge labels n, e, s, w (standing for north, east, south, west). The parameter of $D(W)$ is x, where $x = 2l + 5$ and l is at least the largest edge label in $\mathcal{W}$.

Every tiling $\mathcal{T}$ with $\mathcal{W}$ corresponds directly to a packing $\mathcal{P}$ with $\mathcal{W}$-polyominoes, see Figure 3(a) and 3(b) for an illustration. We have that $\mathcal{P}$ is aperiodic and

$$\text{density}(\mathcal{P}) = \frac{10x - 21}{x^2} = \frac{10}{2\ell + 5} - \frac{21}{(2\ell + 5)^2}. \tag{1}$$

To see that the density of $\mathcal{P}$ is $(10x - 21)/x^2$, observe that the region covered by $\mathcal{P}$ is a periodic set with period x, see Figure 3(c).

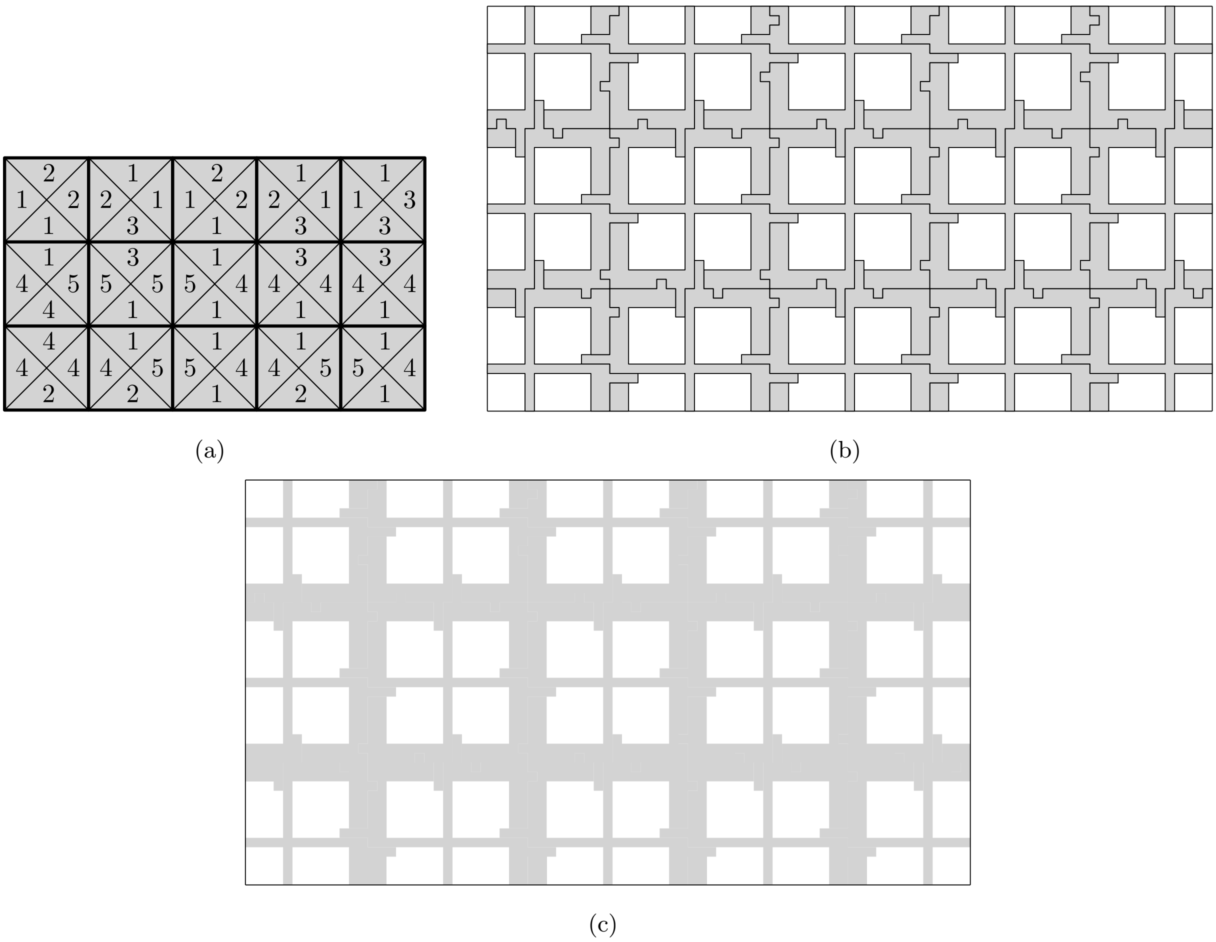

Figure 3: (a) A fragment of a tiling $\mathcal{T}$ with Wang tiles $\mathcal{W}$. (b) The corresponding fragment of a corresponding packing $\mathcal{P}$ with polyominoes $\mathcal{D}(\mathcal{W})$. (c) The region covered by $\mathcal{P}$.

A *difference set of* $\{0, \ldots, q-1\}$ is a subset S of $\{0, \ldots, q-1\}$ such that for every $i \in \{1, \ldots, q-1\}$, there is exactly one ordered pair $(s, s') \in S \times S$ with $i \equiv s - s' \mod q$. Singer's theorem [14] implies that if n is a prime power then there is a difference set of size $n+1$ in $\{0, 1, \ldots, n^2 + n\}$. For example, when $n = 2$, $\{1, 2, 4\}$ is a difference set in $\{0, 1, \ldots, 6\}$.

3 Proofs

Proof of Theorem 1.

Consider any set of polyominoes $\mathcal{D}$ and let $\mathcal{P}$ be any clumsy packing of the plane with $\mathcal{D}$, i.e., density($\mathcal{P}$) = clum($\mathcal{D}$). Such a $\mathcal{P}$ exists by Theorem 6 (see Appendix). For any fixed ϵ we have to construct a periodic packing P_ϵ whose density differs from clum($\mathcal{D}$) by at most ϵ. Let r be the smallest number such that a copy of every polyomino from $\mathcal{D}$ is contained in B_r. Fix $n_0 \geq 8r/\epsilon$ and consider $S := B_{n_0}$. Note that we could assume that $\epsilon \leq 1$ because any packing has density at most 1, hence $n_0 \geq 8r$. Assume, without loss of generality, that $\mathcal{P}$ has the smallest density in S among all $n_0 \times n_0$ squares. Let $\mathcal{P}_0$ be the set of polyominoes from $\mathcal{P}$ that are completely contained in S, let d be the density of $\mathcal{P}_0$ in S. Then the density of $\mathcal{P}$ in S is at least d. Thus,

$$\text{clum}(\mathcal{D}) = \text{density}(\mathcal{P}) \geq d. \tag{2}$$

We shall construct a periodic clumsy packing $\mathcal{P}_\epsilon$ of the plane as follows. First, take the copies of S to tile the plane, and consider the corresponding copies of $\mathcal{P}_0$: $\mathcal{P}' = \{D + (in_0, jn_0) : D \in \mathcal{P}_0,\ i, j \in \mathbb{Z}\}$. We see that $\mathcal{P}'$ is not necessarily a packing. If D is a copy of a polyomino from $\mathcal{D}$ and D does not overlap with any polyomino from $\mathcal{P}'$, add $\{D + (in_0, jn_0) :\ D \in \mathcal{P}_0,\ i, j \in \mathbb{Z}\}$, the set of all appropriate translates of D to $\mathcal{P}'$. Continue in this manner until no copy of D can be added and call the resulting packing $\mathcal{P}_\epsilon$. Note that we have to consider only those copies of a polyomino from $\mathcal{D}$ with a non-empty intersection with S. Since this set is finite, the above process ends after finitely many steps.

We have that S in $\mathcal{P}_\epsilon$ contains the polyominoes from $\mathcal{P}_0$ and parts of the added polyominoes. Each added polyomino is at most $2r - 1$ cells away from the border of S since otherwise one could have added that polyomino to $\mathcal{P}$. Therefore, the added polyominoes contribute at most $4(2r-1)/n_0$ to the density. Thus, using (2), density($\mathcal{P}_\epsilon$) $\leq d + 4(2r-1)/n_0 \leq$ clum($\mathcal{D}$) $+ 4(2r-1)/n_0 \leq$ clum($\mathcal{D}$) $+ \epsilon$. □

Proof of Theorem 2.

Let $\mathcal{D} = \mathcal{D}(\mathcal{W}) \cup \{B\}$, where $\mathcal{D}(\mathcal{W}) = \mathcal{D}(\mathcal{W}, x)$ as defined in Section 2 corresponding to 13 Wang tiles from Figure 1, $x = 2l + 5$, and B is a polyomino corresponding to the $l \times l$ square. Recall that these Wang tiles admit only aperiodic tilings. We say that a copy of B is a bad polyomino and a copy of $D(W_i)$ is a $\mathcal{W}$-polyomino. We denote the set of packings with $\mathcal{D}$ using only polyominoes from $\mathcal{D}(\mathcal{W}, x)$, by $\mathcal{P}_{\mathcal{W}}$.

Idea of the Proof: We shall assume that $\mathcal{P}$ is a periodic clumsy packing with $\mathcal{D}$. If the bad polyomino is not present in $\mathcal{P}$, we shall argue that $\mathcal{P}$ corresponds to a Wang tiling and must be aperiodic, a contradiction. If $\mathcal{P}$ contains a bad polyomino, then it contains a bad polyomino in each large enough square since the packing is periodic. The bad polyominoes contribute a lot to the density of the packing. So, we shall be able to show that in this case the density is strictly greater than the density of a packing with $\mathcal{D}$ using only $\mathcal{D}(\mathcal{W})$.

Let $l = 330$. If a bad polyomino is contained in the region determined by a $\mathcal{W}$-polyomino $D(W)$ as in Figure 4, then we say that the bad polyomino is **secluded** in $D(W)$. We say that a $\mathcal{W}$-polyomino **latches** with another $\mathcal{W}$-polyomino if these are disjoint and adjacent via a side corresponding to the same label of the Wang tiles, see Figure 4.

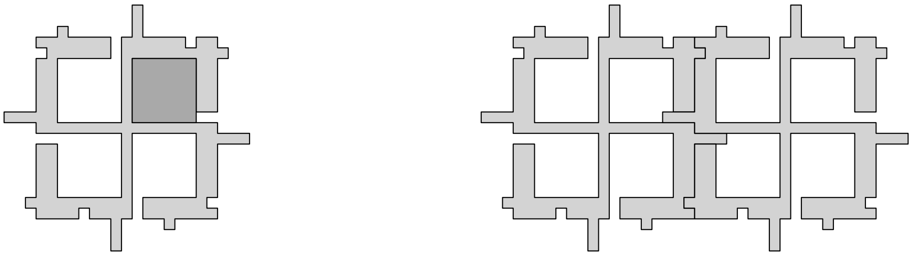

Figure 4: Bad polyomino secluded in a $\mathcal{W}$-polyomino and two latched $\mathcal{W}$-polyominoes.

Observe that in a packing a $\mathcal{W}$-polyomino latches with four $\mathcal{W}$-polyominoes from the top, bottom, left, and right if and only if it has no bad polyomino secluded in itself.
Let d_0 be the density of a packing with $\mathcal{D}$ that uses only $\mathcal{W}$-polyominoes. Recall, see (1), that $d_0 = \frac{10}{2\ell+5} - \frac{21}{(2\ell+5)^2}$. So, $d_0 < \frac{10}{2\ell+5}$.

Case 1 $\mathcal{P}$ does not use B.
In this case, we see that each polyomino in $\mathcal{P}$ is a $\mathcal{W}$-polyomino that is latched from all four sides with $\mathcal{W}$-polyominoes. Thus $\mathcal{P}$ corresponds to the Wang tiling of the plane that is aperiodic. So $\mathcal{P}$ itself is aperiodic, a contradiction.

Case 2 $\mathcal{P}$ uses B.
In this case we shall demonstrate that the density of $\mathcal{P}$ is strictly greater than d_0. Consider a square S with side length $n_0 \cdot 3xk$, where k is a large enough integer and n_0 is the period of $\mathcal{P}$. Split S into $x \times x$ squares and call the set of these $\mathcal{S}$. We call a square $Q \in \mathcal{S}$ a boundary square if it is adjacent to the boundary of S. There are at most $12kn_0 - 4$ boundary squares in $\mathcal{S}$. We call a non-boundary square $Q \in \mathcal{S}$ good if a $3x \times 3x$ square centered at Q does not intersect any bad polyomino. All other squares in $\mathcal{S}$ are called bad.

First observe that $\mathcal{P}$ has density at least d_0 in each good square of $\mathcal{S}$. To see that, let Q be a good square and Q' be the $3x \times 3x$ square centered at Q. Then Q must be intersected by some $\mathcal{W}$-polyomino $D(W)$ that has no bad polyomino secluded in it. Thus $D(W)$ is latched with $\mathcal{W}$-polyominoes from four sides. Assume, without loss of generality, that the center of $D(W)$ is closest to the bottom right corner of Q as shown in Figure 5. If $D(W_1)$ and $D(W_2)$ are polyominoes latched with $D(W)$ from the top and left, then $D(W), D(W_1), D(W_2)$ are contained completely inside Q'. Thus, $D(W_1)$ and $D(W_2)$ have no bad polyominoes secluded in them, so, in particular, $D(W_1)$ is latched with a $\mathcal{W}$-polyomino $D(W_3)$ on the left that is also completely contained in Q', and also does not have a bad polyomino secluded in it. So, we observe that Q in $\mathcal{P}$ corresponds to a fragment of a packing from $\mathcal{P}_{\mathcal{W}}$ and contains exactly d_0x^2 occupied cells.

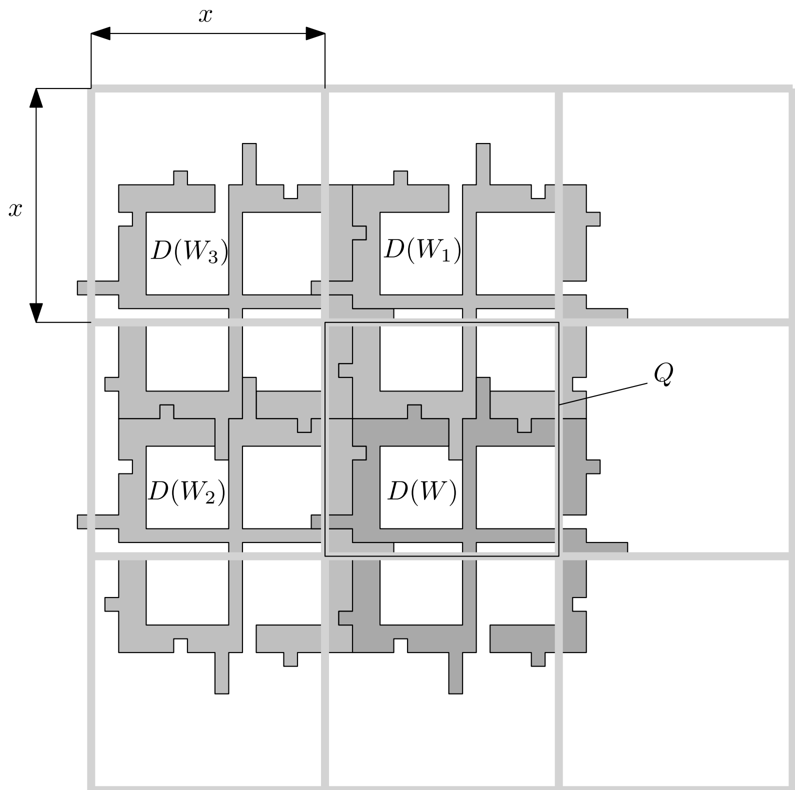

Figure 5: A good square Q centered at a $3x \times 3x$ square not intersecting any bad polyomino.

Second, observe that $\mathcal{P}$ has density at least $d_0 + \epsilon$ in the region covered by bad squares of $\mathcal{S}$, for some positive ϵ. This follows from the fact that each copy of B makes at most 16 squares in $\mathcal{S}$ bad, so the density of $\mathcal{P}$ in the region covered by bad squares is at least $l^2/16x^2$. Thus, density of $\mathcal{P}$ in the region covered by bad squares is at least

$$\frac{l^2}{16x^2} = \frac{l^2}{16(2l+5)^2} > \frac{10}{2l+5} + \epsilon > d_0 + \epsilon,$$

when $l \geq 330$ and ϵ is some positive constant.

Now, if b is the number of bad squares in $\mathcal{S}$, then, since the number of boundary squares is at most $12kn_0$, the number of good squares is at least $(3kn_0)^2 - 12kn_0 - b$. Therefore the density of $\mathcal{P}$ in S is at least

$$d \geq \frac{bx^2(d_0+\epsilon) + ((3kn_0)^2 - 12kn_0 - b)x^2 d_0}{(3xkn_0)^2} = \frac{b\epsilon'}{(3kn_0)^2} + d_0 - \frac{4d_0}{3kn_0},$$

where the first term in the nominator corresponds to the area occupied by $\mathcal{P}$ in bad squares, the second term gives a bound on such an area in good squares, and the denominator is the total area of S. Since each $n_0 \times n_0$ square intersects non-trivially a copy of B, $b \geq (3xk)^2$. Thus

$$d \geq \frac{k^2x^2\epsilon}{k^2n_0^2} + d_0 - \frac{4d_0}{3kn_0} \geq \frac{x^2\epsilon}{n_0^2} + d_0 - \left(\frac{4d_0}{3n_0}\right)\frac{1}{k} > d_0 + \frac{x^2\epsilon}{2n_0^2},$$

if k is chosen sufficiently large, for example $k > \frac{n_0^2}{x^2\epsilon}\frac{8d_0}{3n_0}$. So, we have that d is strictly larger than d_0, a contradiction to the fact that $d = \text{clum}(\mathcal{D}) \leq d_0$.

□

Proof of Theorem 3. This proof is very similar to the proof of Theorem 2 above. So, we sketch only the main parts of the proof here.

Berger [1] proved that the question whether a given finite set $\mathcal{W}$ of Wang tiles tiles the plane is undecidable. We reduce this *tiling problem* to the problem of deciding whether a given set $\mathcal{D}$ of polyominoes has clumsiness at most a given rational number d. That is, given any finite set $\mathcal{W}$ of Wang tiles, we construct a set $\mathcal{D}$ of polyominoes and a rational number d, such that $\text{clum}(\mathcal{D}) \leq d$ if and only if $\mathcal{W}$ tiles the plane. This implies that the question whether a given finite set of polyominoes has clumsiness at most a given rational number is undecidable, too.

So let $\mathcal{W}$ be any finite set of Wang tiles. We again consider the set $\mathcal{D} = \mathcal{D}(\mathcal{W}, x) \cup \{B\}$, where $\mathcal{D}(\mathcal{W}, x)$ as defined in Section 2 with $x = 2l+5$ and $l \geq \max\{330, q\}$, where q is the largest edge label in $\mathcal{W}$. Let B be the so-called bad polyomino corresponding to the $l \times l$ square defined as before. Recall that every tiling of the plane with the Wang tiles $\mathcal{W}$ directly corresponds to a packing with $\mathcal{D}$ using only polyominoes from $\mathcal{D}(\mathcal{W}, x)$ whose density is $d := (10x-21)/x^2 \in \mathbb{Q}$. Let $\mathcal{P}_{\mathcal{W}}$ denote the set of packings with $\mathcal{D}$ using only polyominoes from $\mathcal{D}(\mathcal{W}, x)$. We claim that $\text{clum}(\mathcal{D}) = d$ if the Wang tiles $\mathcal{W}$ tile the plane and $\text{clum}(\mathcal{D}) > d$ otherwise.

So fix $\mathcal{P}$ to be any clumsy packing with $\mathcal{D}$. Consider any $n_0 \times n_0$ square Q in the plane, where n_0 is any multiple of x. In the proof of Theorem 2 we argued that if Q intersects no bad polyomino, then the $(n_0 - x) \times (n_0 - x)$ square concentric with Q corresponds to a fragment of a packing from $\mathcal{P}_{\mathcal{W}}$ and it occupies the area exactly equal to $d \cdot n_0^2$. Hence if for every such $n_0 \in \mathbb{N}$ such a square Q exists, then for every $n \in \mathbb{N}$ the $n \times n$ square can be tiled by the Wang tiles $\mathcal{W}$, which is well-known to be equivalent to $\mathcal{W}$ tiling the entire plane.

On the other hand if there exists $n_0 \in \mathbb{N}$ such that every $n_0 \times n_0$ square intersect at least one bad polyomino, then the same argumentation as in the proof of Theorem 2 shows that $\text{density}(\mathcal{P}) > d$ and thus $\text{clum}(\mathcal{D}) > d$. □

The next Lemma will be used in the proof of Theorem 4.

Lemma 1. *Every polyomino of size k intersects at most $k^2 - k + 1$ copies of itself, while if the polyomino is connected, it intersects at most $k^2 - (\lfloor (k-1)/2 \rfloor^2 + \lceil (k-1)/2 \rceil^2)$ copies of itself. Moreover, both bounds are best possible.*

Proof. For a polyomino D of size k, we write the set of coordinates $C = C(D) \subseteq \mathbb{Z}^2$ of the cells of D, say by lower-left corner of each cell. The number of copies of D intersecting D is equal to the number of distinct vectors $(i, j) \in \mathbb{Z}^2$ that correspond to a difference of some two elements in C, i.e., $|C - C| = |\{x - y : x, y \in C\}|$. We see that $|C - C| \leq |C|^2 - |C| + 1 = k^2 - k + 1$, because each pair $(x, y) \in C^2$ gives a difference $x - y$, and $|C|$ pairs $(x, x) \in C^2$ give the same difference. This provides the first claim of the Lemma.

Now, we shall assume that D is connected. In this case, we shall observe that many more distances in $C - C$ are represented by more than one pair (x, y), $x, y \in C$. In particular, the key observation is that if $x, y \in C$ and $x', y' \in C$, for $x' = x + (1, 0)$, $y' = y + (1, 0)$ then $v = x - y$ is represented by both the pair (x, y) and the pair (x', y'). So, we shall analyze such double-counts by introducing an auxiliary tree T and its special sets of vertices A and B below.

Let G be the directed graph with vertex set C and directed edges of the form $(x, x + (1, 0))$ or $(x, x + (0, 1))$. Let T be a spanning directed tree in G. Let $A = \{x \in C : (x, x + (1, 0)) \in E(T)\}$, $\quad B = \{x \in C : (x, x + (0, 1)) \in E(T)\}$. We can visualize T as a tree, with vertices corresponding to the cells of D, where A provides those cells of D that have a directed edge in T to the right, B provides the cells with a directed edge in T going straight up, $C - A - B$ are the cells of D with no out-edges in T.

For $v \in C - C$, we shall build a maximal sub forest T_v of T:

$$V(T_v) = \{x \in C : x + v \in C\}, \quad E(T_v) = \{(x, x') \in E(T) : \ (x + v, x' + v) \in E(T)\}.$$

See Figure 6 for an example of such a tree.

One way to visualize T_v is to start with a single vertex $x \in C$ such that $x + v \in C$ and move the vector v with endpoints x and $x + v$ one unit along the edges of T (if this movement is possible for $x + v$ in T as well), to create another vertex and edge incident to x in T_v. Continuing in this manner as long as possible gives a component of T_v. By induction on the number of vertices one can prove that the number of vertices of out-degree 2 in T_v is strictly less than the number of vertices of out-degree 0. This shows that

$$|\{(x, y) \in (A \cap B)^2 : v = y - x\}| < |\{(x, y) \in C^2 \setminus (A^2 \cup B^2) : v = y - x\}|.$$

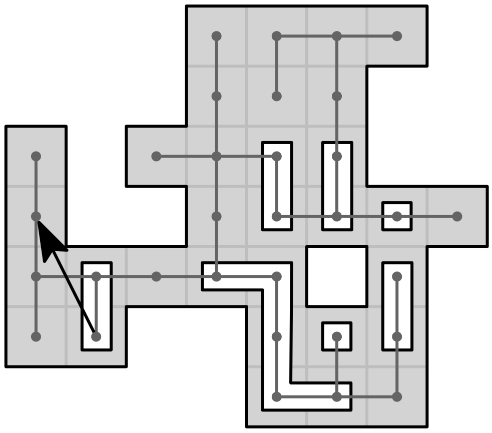

Figure 6: A connected polyomino D together with a spanning directed tree T (gray edges). The connected components of the sub forest T_v w.r.t. the distance $v = (-1, 2) \in C - C$ (black arrow) is indicated by the white regions.

Now, we are ready to make the final calculation:

$$\begin{aligned}
|C - C| &= \sum_{v \in C-C} 1 \\
&\le \sum_{v \in C-C} |\{(x,y) \in C^2 \setminus (A^2 \cup B^2) : v = y - x\}| - |\{(x,y) \in (A \cap B)^2 : \ v = y - x\}| \\
&= |\{(x,y) \in C^2 \setminus (A^2 \cup B^2\}| - |\{(x,y) \in (A \cap B)^2\}| \\
&= |C^2 \setminus (A^2 \cup B^2)| - |(A \cap B)^2| \\
&= \big(|C^2| - |A^2| - |B^2| + |(A \cap B)^2|\big) - |(A \cap B)^2| \\
&= k^2 - |A|^2 - |B|^2.
\end{aligned}$$

Since $|A| + |B| = |E(T)| = k - 1$, $|A|^2 + |B|^2 \ge \lfloor (k-1)/2 \rfloor^2 + \lceil (k-1)/2 \rceil^2$. Therefore, the total number of distinct copies of D intersecting D is at most $k^2 - (\lfloor (k-1)/2 \rfloor^2 + \lceil (k-1)/2 \rceil^2)$. $\square$

To see that the bounds stated are attained, see Figure 7.

Proof of Theorem 4. First we shall show the lower bound on the density of any packing with $\{D\}$, where D is a polyomino of area k. Let $f(D)$ be the number of copies of D intersecting D and r be the smallest integer such that a copy of D is contained in B_r. Let $\mathcal{P}$ be a packing with $\{D\}$ and $\mathcal{C}$ be the set of all copies of D. Let n be some large enough integer. Let

$$\mathcal{P}_n = \{D' \in \mathcal{P} : D' \subseteq B_n\} \text{ and } X_n = \{D' \in \mathcal{C} : D' \subseteq B_{n-4r}\}.$$

So, for each $D' \in X_n$, D' is contained in B_n and all copies of D intersecting D' are also contained in B_n.

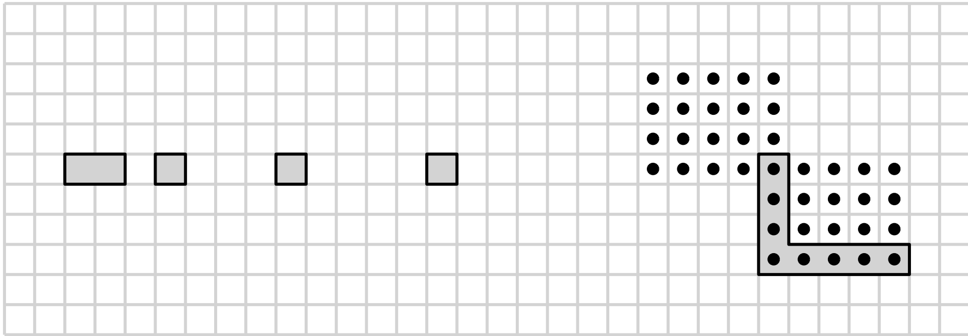

Figure 7: **Left:** A polyomino of size $k = 5$ that intersects exactly $k^2 - k + 1$ copies of itself (including itself). It is defined based on the set $\{1, 2, 4, 8, 13\}$, whose elements have exactly $k^2 - k + 1$ distinct differences. **Right:** A connected polyomino of size $k = 2r$ that intersects exactly $k^2 - \lfloor (k-1)/2 \rfloor^2 - \lceil (k-1)/2 \rceil^2 = 2r(r+1) - 1$ copies of itself (including itself). The points mark the position of the topmost cell of each intersected copy.

Using the fact that each polyomino in X_n intersects non-trivially a polyomino from $\mathcal{P}_n$ and each polyomino intersects $f(D)$ copies of itself, we have that $|X_n| \leq f(D)|\mathcal{P}_n|$. Thus $|\mathcal{P}_n| \geq \frac{|X_n|}{f(D)}$. Therefore, the density of $\mathcal{P}$ is at least

$$\limsup_{n \to \infty} \frac{k|\mathcal{P}_n|}{|B_n|} \geq \limsup_{n \to \infty} \frac{k|X_n|}{f(D)|B_n|} = \frac{k}{f(D)},$$

since $|X_n|/|B_n| \to 1$ as $n \to \infty$. Now, using Lemma 1, we see that for any polyomino D of area k, $f(D) \leq k^2 - k + 1$ and for any connected polyomino D of area k, $f(D) \leq k^2 - \lfloor (k-1)/2 \rfloor^2 - \lceil (k-1)/2 \rceil^2$. Thus, both lower bounds follow.

Next we shall show that both bounds are tight. In case of connected polyominoes, a packing given in Figure 8 has density exactly $\frac{k}{k^2 - \lfloor (k-1)/2 \rfloor^2 - \lceil (k-1)/2 \rceil^2}$. For the general case, the lower bound is attained for k such that $k-1$ is a prime power, otherwise, it is asymptotically attained as k grows.

If $k-1$ is a prime power, let S be a difference set of size k in $\{0, \ldots, k^2 - k\}$. Let $q = k^2 - k + 1$. Let a polyomino D have cells whose left lower corners have coordinates $\{(s, 0) : s \in S\}$. Let $\mathcal{P}$ be the packing of D with the following set of copies of D: $\{D + (qi, j) : \ i, j \in \mathbb{Z}\}$. I.e., we take the copies of D and put them in each horizontal strip next to each other. We claim that $\mathcal{P}$ is a packing, i.e., adding any copy D' of D will non-trivially intersect a copy of D from $\mathcal{P}$. Take, without loss of generality $D' = D + (t, 0)$ for some $t \in \mathbb{Z}$. Let $s, s' \in S$ such that $s' - s \equiv t \pmod{q}$. Then $(s + t, 0) \in D'$, on the other hand $s + t = s' + qi$ for some $i \in \mathbb{Z}$ and therefore $(s + t, 0) \in (D + (qi, 0))$ and indeed D' intersects a polyomino from $\mathcal{P}$. Thus $\mathcal{P}$ is a packing with density $\frac{k}{k^2 - k + 1}$.

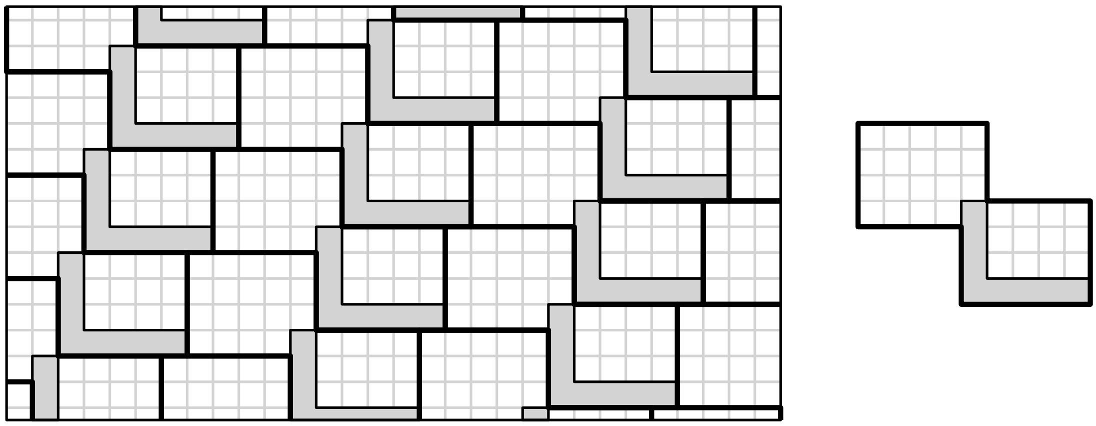

Figure 8: A packing with a single connected polyomino of area $k = 8$ that has density $8/19 = k/(k^2 - \lfloor(k-1)/2\rfloor^2 - \lceil(k-1)/2\rceil^2)$.

If $k-1$ is not a prime power, let $k'-1$ be the largest prime power not exceeding $k-1$. Let S be a k-element subset of $\{0, \ldots, k'^2 - k'\}$ containing the difference set of size k'. Construct a polyomino D and a packing $\mathcal{P}$ based on S the same way as above. Then $\mathcal{P}$ is a packing with density $\frac{k}{k'^2-k'+1}$ that approaches $\frac{k}{k^2-k+1}$ for large k.

□

4 Conclusion

We have shown that there are polyomino sets with all clumsy packings aperiodic. We proved that the question whether a given set of polyominoes has clumsiness at most a given rational number is undecidable. We calculated tight lower bounds on the clumsiness of the packings using a copy of a single polyomino. Finally, we have shown that there is always a periodic packing arbitrarily close in density to a clumsy packing. There are many questions that remained unanswered. Our aperiodic packings correspond to a periodic region in the plane. Is it possible to construct a set of tiles such that any clumsy packing is aperiodic and the covered region is also aperiodic? We believe that the answer to this question is "yes". Consider the proof of Theorem 2 with $\mathcal{D} = \mathcal{D}(\mathcal{W}) \cup \{B\}$ and make a slight modification to polyominoes $\mathcal{D}(\mathcal{W})$ by moving the long key to a distance from the center depending on the label of the corresponding edge of a Wang tile, see Figure 9 for comparison. Clearly, the packing using only $\mathcal{D}(\mathcal{W})$ is aperiodic and the region of the plane covered by this packing is also aperiodic. It remains to verify that the density of a packing involving a positive fraction of polyomino B is strictly larger than the density of a packing using only $\mathcal{D}(\mathcal{W})$.

What is the smallest possible density of a packing with a set of polyominoes obtained by rotating a single polyomino of area k by $j\pi/2,\ j \in \mathbb{Z}$? How small can one make the polyominoes in a set whose clumsy packings are aperiodic? What is the smallest clumsiness of a set consisting

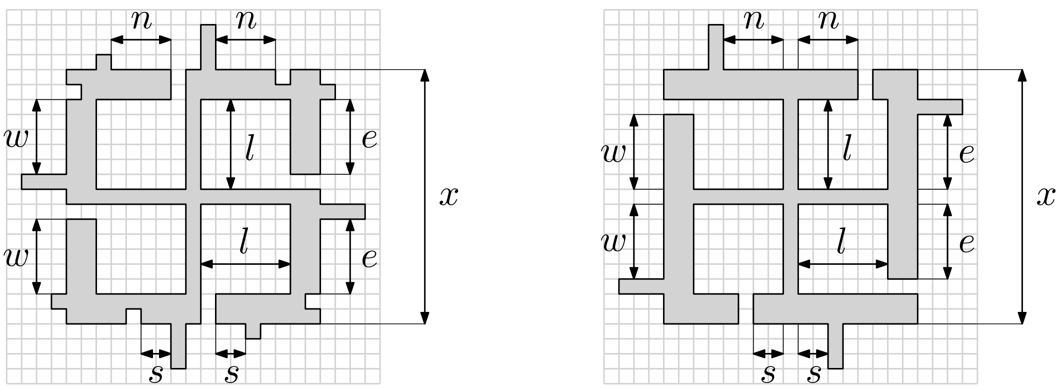

Figure 9: Left: Polyominoes as defined in Section 2. Right: Alternative definition for which a packing covers an aperiodic region of the plane.

of polyominoes of order k?

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5 Appendix

Theorem 6. *For every set of polyominoes $\mathcal{D}$ there is a clumsy packing of the plane with $\mathcal{D}$.*

Proof of Theorem 6.
Let $\mathcal{D}$ be a set of polyominoes and r be the smallest integer such that a copy of every polyomino in $\mathcal{D}$ is contained in B_r. Let d be the infimum of densities of packings with $\mathcal{D}$. We see that for each n there is a packing of the plane with $\mathcal{D}$ with density at most $d + 1/n$. We shall show that there is a packing with $\mathcal{D}$ of density equal to d. Here, by a restriction of a set of polyominoes $\mathcal{P}$ to a set S of cells we mean the set $\{P \in \mathcal{P} : \ P \subseteq S\}$.

Take a packing $\mathcal{P}$ of the plane with density at most $d + 1/n$ and consider its restrictions to $n \times n$ squares. Among those, choose a set of smallest density. Assume that this density is at least $d + 1/n + \epsilon$, for some positive ϵ and consider large enough m divisible by n. Since B_m contains $(m/n)^2$ disjoint copies of B_n, the density of $\mathcal{P}$ in B_m is at least $(m/n)^2(dn^2 + n + \epsilon n^2)/m^2 = d + 1/n + \epsilon$, a contradiction to the fact that the density of $\mathcal{P}$ is at most $d + 1/n$. Thus there is a restriction $\mathcal{P}'$ of $\mathcal{P}$ to some $n \times n$ square with density at most $d + 1/n$. We have that $\mathcal{P}'$ is not necessarily a packing of the square, but becomes one after adding some polyominoes at most $2r$-far from the boundary of the $n \times n$ square. The total number of added cells is at most $8rn$, thus we get a packing of $n \times n$ square with density at most $(dn^2 + n + 8rn)/n^2 \leq d + 9r/n$.

For every positive integer n consider a packing $\mathcal{P}_n$ of B_n with $\mathcal{D}$ with smallest density among all packings of B_n. Such a packing exists since B_n is finite. This way we get an infinite sequence $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \ldots$ of packings with $\mathcal{D}$. Note that for every large enough $n \in \mathbb{N}$ the density of $\mathcal{P}_n$ is $d + q$, where $q \leq 9r/n$ from the above observation. Next we show that for each n and $n' < n$, the density of P_n restricted to $B_{n'}$ is at most $d+26r/n'$. If this does not hold, then consider P_n, delete all polyominoes intersecting B'_n, add $P_{n'}$ and add all polyominoes that could be added (those are at most $2r$-far from the boundary of $B_{n'}$ and thus occupy at most $16rn'$ cells). Assuming that the density of P_n is $d+q$, we get a packing of B_n with the number of occupied cells at most $(d+q)n^2-(d+26r/n')n'^2+(d+9r/n')n'^2+16rn' = (d+q)n^2-26rn'+9rn'+16rn' = (d+q)n^2-(26-25)rn' < (d+q)n^2$. This contradicts the minimality of P_n.

Therefore, for every $m > n$ and an absolute constant c, we have

$$\Big|\big(\bigcup_{D\in\mathcal{P}_m} D\big)\cap B_n\Big| \le \Big|\bigcup_{D\in\mathcal{P}_n} D\Big| + crn. \tag{3}$$

Next we shall define a sequence $Q_0, Q_1, \dots$ of sets of disjoint copies of polyominoes from $\mathcal{D}$ with the following properties:

(i) $Q_0 = \emptyset$;

(ii) $Q_{n-1} \subseteq Q_n$;

(iii) Q_n is a restriction of $\mathcal{P}_m$ to B_n for infinitely many $m \ge n$.

We show that Q_n exists by an inductive argument starting with an obvious case Q_0. Assuming that Q_{n-1} exists, we have $Q_{n-1} \subset \mathcal{P}_m$ for $m \in M$, M is an infinite set. Consider restrictions of P_m, $m \in M$ to B_n. There are only finitely many of those, so one of the restrictions, call it Q_n, appears in $P_{m'}$, $m' \in M'$, where M' is an infinite subset of M.

Finally, we shall define $\mathcal{P}$, a desired clumsy packing: $\mathcal{P} = \bigcup_{i=0}^{\infty} Q_i$. Since no polyomino could be added to Q_i in B_{i-2r}, we see that $\mathcal{P}$ is a packing with $\mathcal{D}$. To verify that $\mathcal{P}$ is indeed a clumsy packing with $\mathcal{D}$ we calculate its density. For $n \in \mathbb{N}$ let $m(n) > n$ be any integer such that Q_n is a restriction of $\mathcal{P}_{m(n)}$ to B_n. Then, using (3),

$$\begin{aligned}\text{density}(\mathcal{P}) &= \limsup_{n\to\infty} \frac{\left|\left(\bigcup_{D\in\mathcal{P}} D\right)\cap B_n\right|}{|B_n|} = \limsup_{n\to\infty} \frac{\left|\left(\bigcup_{D\in\mathcal{P}_{m(n)}} D\right)\cap B_n\right|}{|B_n|} \\ &\le \limsup_{n\to\infty} \frac{\left|\left(\bigcup_{D\in\mathcal{P}_n} D\right)\right| + crn}{|B_n|} \le \limsup_{n\to\infty} (dn^2+9rn+crn)/n^2 = d.\end{aligned}$$

□