A note on adjacent vertex distinguishing colorings number of graphs

M. Axenovich * J. Harant † J. Przybyło ‡ R. Soták § M. Voigt ¶

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Abstract

For an assignment of numbers to the vertices of a graph, let $S[u]$ be the sum of the labels of all the vertices in the closed neighborhood of u, for a vertex u. Such an assignment is called *closed distinguishing* if $S[u] \neq S[v]$ for any two adjacent vertices u and v unless the closed neighborhoods of u and v coincide. In this note we investigate $\mathrm{dis}[G]$, the smallest integer k such that there is a closed distinguishing labeling of G using labels from $\{1, \ldots, k\}$. We prove that $\mathrm{dis}[G] \leq \Delta^2 - \Delta + 1$, where Δ is the maximum degree of G. This result is sharp. We also consider a list-version of the function $\mathrm{dis}[G]$ and give a number of related results.

1 Introduction

One of the important tasks in network studies is to be able to identify and distinguish its elements, e.g. vertices, using local substructures and small labels. Among the multiple ways to achieve this, one of the most natural ones is to distinguish vertices by the sums of labels in their neighborhoods.

Formally, for a graph G, the open neighborhood of a vertex v is denoted $N(v)$ and the closed neighborhood is denoted $N[v]$. We say that an assignment of numbers (labels) to the vertices of G is *closed distinguishing* if the sum of labels of the vertices in $N[v]$ differs from the sum of labels of the vertices in $N[u]$ for any adjacent vertices u and v unless $N[u] = N[v]$. Let $\mathrm{dis}[G]$ be the smallest integer k such that there is a closed distinguishing assignment for G using integers from the set $\{1, \ldots, k\}$. Define also $\mathrm{dis}(G)$ using $N(u)$ instead of $N[u]$ and call the corresponding coloring *open distinguishing*. Note that both values $\mathrm{dis}[G]$ and $\mathrm{dis}(G)$

*Karlsruher Institut für Technologie, Germany maria.aksenovich@kit.edu.

†Technical University of Ilmenau, Germany jochen.harant@tu-ilmenau.de

‡AGH University of Science and Technology, al. A. Mickiewicza 30, 30-059 Krakow, Poland przybylo@wms.mat.agh.edu.pl

§Pavol Jozef Šafárik University in Košce, Slovakia roman.sotak@upjs.sk

¶Dresden University of Applied Sciences, Germany mvoigt@informatik.htw-dresden.de

exist because an assignment of n distinct powers of 2 to the n-vertex graph gives distinct sums for any two subsets of vertices. A better upper bound on $\text{dis}(G)$ might be derived from Theorem 1. This and other results on $\text{dis}(G)$ in planar graphs can be found in [9, 16], where in particular it is conjectured that $\text{dis}(G)$ is at most the chromatic number $\chi(G)$ of G.

Theorem 1 (S. Norin, [16])**.**
Let G be a graph with chromatic number r and coloring number k. Let $n_1, ..., n_r$ be pairwise coprime integers with $n_i \geq k$ for $i = 1, ..., r$. Then $\text{dis}(G) \leq n_1 n_2 ... n_r$.

While the asymptotic behavior of $\text{dis}(G)$ remains a widely open problem, we focus on the $\text{dis}[G]$ function in this note. It is worth mentioning that vertex-distinguishing labelings have already been thoroughly investigated in the case of planar graphs. By taking $n_1 = 7$, $n_2 = 8$, $n_3 = 9$, and $n_4 = 11$ in Theorem 1, it follows that $\text{dis}(G) \leq 5544$ for a planar graph G. In [16], this bound is improved to 468. Moreover, it is shown there that $\text{dis}(G) \leq 36$ for a 3-colorable planar graph, that $\text{dis}(G) \leq 4$ for a planar graph of girth at least 13, and that $\text{dis}(G) \leq 2$ if G is a tree. The large body of research in this area uses not only the labels on vertices but also on the edges and the faces of a graph. One of the fascinating conjectures in the area is so-called $1-2-3$ Conjecture by Karoński, Łuczak, and Thomason posed in [21]. It claims that it is sufficient to use integers $1, 2$ and 3 in order to label the *edges* of any non-trivial (of order ≥ 3) connected graph so that adjacent vertices meet distinct sums of their incident labels. This conjecture is true e.g. for 3-colorable graphs, see [21], and it is known that labels chosen from the set $\{1, 2, \ldots, 5\}$ are sufficient, as shown by Kalkowski, Karoński, and Pfender in [20]; see also [26], [4], and [5] for earlier results. A number of variations of distinguishing labelings of graphs have also been considered, see for example [1, 2, 6, 7, 8, 10, 12, 13, 14, 17, 19, 22, 23, 24, 27]. See also a survey on graph labelings by Galian [15].

In this note we initiate the study of $\text{dis}[G]$, provide the sharp upper bound on this parameter in terms of the maximum degree of a graph and several related results. Our main theorem is stated in a general setting of list-colorings. Assume that every vertex is endowed with a list $L(v) \subseteq \mathbb{R}$ of available labels. Let $\text{dis}_\ell[G]$ ($\text{dis}_\ell(G)$) be the smallest k such that for every list assignment with $|L(v)| \geq k$ for all $v \in V(G)$ there is a closed (open) distinguishing assignment giving every vertex v a label from its list and $\chi_\ell(G)$ is the smallest integer k such that for every list assignment L with $|L(v)| \geq k$ there is a proper vertex coloring c with $c(v) \in L(v)$ for all $v \in V$. Trivially, for complete graph K_n we have $\text{dis}_\ell[K_n] = \text{dis}[K_n] = 1$ and $\text{dis}_\ell(K_n) = \text{dis}(K_n) = \chi_\ell(K_n) = n$.

Below we formulate our results. We prove them in Section 2.

Theorem 2. *Let $\Delta \geq 2$ be an integer and G be a graph with $\Delta(G) = \Delta$. Then*

- $\text{dis}[G] \leq \text{dis}_\ell[G] \leq \Delta^2 - \Delta + 1$.
- *For every Δ, there is a graph G such that* $\text{dis}[G] \geq (1 - o(1))\Delta^2$.
- *There are infinitely many values of Δ for which G might be chosen so that* $\text{dis}[G] = \Delta^2 - \Delta + 1$.

Theorem 3. *Let $\Delta \geq 2$, $k \geq 2$ be integers and G be a graph with the coloring number k and $\Delta(G) = \Delta$. Then* $\mathrm{dis}_\ell(G) \leq (k-1) \cdot \Delta + 1 \leq \Delta^2 + 1$.

The coloring number of a planar graph is at most 6. Hence, we have immediately the following Corollary.

Corollary 4. *If G is a planar graph, then* $\mathrm{dis}_\ell(G) \leq 5 \cdot \Delta + 1$.

If $\Delta \leq 93$ then this result gives a bound that is better than the best bound 468 known so far.

One of the challenging problems in the area is to determine how 'dis' function depends on the chromatic number of a graph. The situation is far from being understood even for bipartite graphs. Let us assign the vertices of one partite set of a bipartite graph G the label 0 and the vertices in the other partite set the label 1. This gives an open distinguishing coloring of G and a closed distinguishing labeling when the vertices in one partite set have degrees at least 2. However, in general, or when the label 0 is not allowed, the situation is not clear. We prove the following.

Theorem 5. *Let G be a bipartite graph with partite sets A and B which is not a star. Let, for $X \in \{A, B\}$, $\Delta_X = \max_{x \in X} d(x)$ and $\delta_{X,2} = \min_{x \in X, d(x) \geq 2} d(x)$, where $d(x)$ denotes the degree of x in G. Then*

$$\mathrm{dis}[G] \leq \min\{c\sqrt{|E(G)|}, \left\lfloor \frac{\Delta_A - 1}{\delta_{B,2} - 1} \right\rfloor + 1, \left\lfloor \frac{\Delta_B - 1}{\delta_{A,2} - 1} \right\rfloor + 1\},$$

where c is some constant. For every cycle C_n of length $n \geq 4$, $\mathrm{dis}_\ell[C_n] = \mathrm{dis}[C_n] = \chi(C_n)$.

In case of trees we provide results for list, modulo p, and ordinary sum-distinguishing numbers.

Theorem 6. *Let $T \neq K_2$ be a tree.*

- *We have* $\mathrm{dis}_\ell[T] \leq 3$ *and* $\mathrm{dis}[T] \leq 2$.
- *If $L(v)$, $v \in V(T)$, are lists of 2 positive numbers each, then T has a closed distinguishing labeling from these lists.*
- *Let $p \geq 4$ be an integer. There exists a labeling $w : V(T) \to \{1, 2, 3\}$ such that* $\sum_{u \in N[x]} w(u) \not\equiv \sum_{v \in N[y]} w(v) \pmod{p}$ *for all edges $xy \in E(T)$.*

We say that a forest is *strong* if it does not have a K_2 component.

Corollary 7. *Let G be a graph whose edges are covered by k induced strong forests. Let $p_1, \ldots, p_k$ be pairwise relatively prime integers, each at least 4. Then* $\mathrm{dis}[G] \leq p_1 p_2 \ldots p_k$.

2 Proofs

Given any vertex-labeling w of a graph $G = (V, E)$ and a vertex $x \in V$, we denote

$$S[x] = \sum_{v \in N[x]} w(v) \quad \text{and} \quad S(x) = \sum_{v \in N(x)} w(v)$$

further on.

2.1 Proof of Theorem 2

Lower Bound

We need to define a special graph $CP(m)$, the *corona of the projective plane*. Let P be the projective plane of order m with a set $V(P)$ of $m^2 + m + 1$ vertices, lines $L_1, \dots, L_{m^2+m+1}$ each containing exactly $m + 1$ vertices from $V(P)$ such that each vertex belongs to exactly $m + 1$ lines and each pair of vertices belongs to some line. Now, we define the vertex set of $CP(m)$ to be the disjoint union of $V(P)$ and $m^2 + m + 1$ disjoint $(m + 1)$-element sets $X_1, \dots, X_{m^2+m+1}$. The edge set of $CP(m)$ is formed by all edges induced by every X_i, $1 \le i \le m^2 + m + 1$ and a perfect matching between X_i and L_i, for each $1 \le i \le m^2 + m + 1$. We call the set $V(P)$, the *core* of $CP(m)$ and the sets $X_1, \dots, X_{m^2+m+1}$, the *spikes* (see Figure 1). Note that G is regular of degree $m + 1$.

Assume first that $\Delta = m + 1$, for a prime m. Thus there is a projective plane of order m.

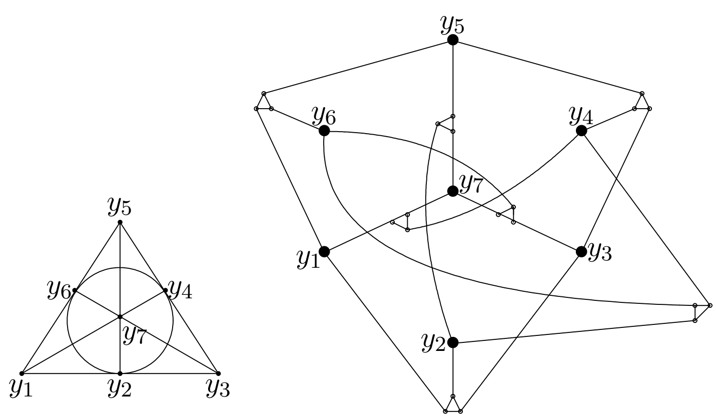

Figure 1: The construction of $CP(m)$ for the Fano plane; $m = 2$

Let $G = CP(m)$ be the corona of the projective plane, a regular graph of degree Δ, and let w be a closed distinguishing labeling of G. For a line L_i of G, any two vertices $x, y \in L_i$ and their respective neighbors $x', y' \in X_i$, we have that $0 \ne S[x'] - S[y'] = w(x) - w(y)$, since all vertices are adjacent in the spike X_i. Thus, any two vertices that belong to a line of P have distinct colors. Since any two vertices belong to a line, the core of G is completely multicolored. Thus the total number of colors required for G is at least $|V(P)| = m^2+m+1$. Since $\Delta(G) = \Delta = m+1$, the number of colors is at least $(\Delta-1)^2+(\Delta-1)+1 = \Delta^2-\Delta+1$.

If $\Delta \neq p+1$, for a prime p, we take the largest appropriate prime m, such that $m+1 \leq \Delta - 1$. It is known, see the paper by Baker et al. [3], that for any sufficiently large integer x, there is a prime number in the interval $[x - x^{0.525}, x]$. The corresponding graph $CP(m)$ requires the number of colors that is at least $m^2+m+1 \geq ((\Delta-2)-(\Delta-2)^{0.525})^2 + (\Delta-2) - (\Delta-2)^{0.525} + 1 = \Delta^2(1-o(1))$. Observe further, that one can construct a graph G' of maximum degree Δ with dis$[G']$ at least as large as the lower bound stated in the theorem and with arbitrarily large number of vertices. This can be done by taking any graph H and identifying one of its vertices with any vertex from the core of the graph $CP(m)$ constructed above. It is easy to see that for the obtained graph G', dis$[G'] \geq$ dis$[CP(m)]$ while $\Delta(G')$ might be chosen to be arbitrary number $\geq m+1$.

Upper Bound

Let G be a graph with maximum degree Δ. Assume that every vertex v is endowed with a list $L(v)$ of $\Delta(\Delta-1)+1$ available labels. Consider an arbitrary ordering $v_1, \ldots, v_n$ of the vertices. We shall associate labels with vertices in this order. Before we start, assign an initial label, $w(v) = 0$ to each vertex $v \in V(G)$ regardless of $0 \in L(v)$.

Let $xy \in E(G)$ be called *bad* for w if $S[x] = S[y]$ and $N[x] \neq N[y]$, otherwise xy is *good*. Our goal is to modify the labeling w in n steps so that there are no bad edges left and $w(u) \in L(u)$ for $u \in V(G)$.

Step 1. Choose $w(v_1) \in L(v_1)$.

Step i**,** $2 \leq i \leq n$**.** After the vertices $v_1, \ldots, v_{i-1}$ have been assigned labels from their lists, we consider v_i and the set $E(v_i)$ of edges not incident to v_i and having exactly one endpoint adjacent to v_i. Note that the edges of $E(v_i)$ are the only edges of G which may change their status (good/bad) if we modify the label of v_i. Indeed, if $e \notin E(v_i)$ then $w(v_i)$ contributes equally to the sums of both end vertices of e or to none of them.

If $e \in E(v_i)$ was good for w after step $i-1$, there is at most one value w_e so that if $w(v_i) = w_e$ then e becomes bad. If an edge e of $E(v_i)$ was bad for w after step $i-1$, it will remain bad after step i if and only if $w(v_i) = w_e = 0$. Since $|E(v_i)| \leq \Delta(\Delta-1)$, there is a choice for $w(v_i) \in L(v_i)$ so that all edges of $E(v_i)$ are good afterwards.

The algorithm stops after the nth step. All edges e that belong to $E(v_i)$ at some step of the algorithm, are good for w. I.e., if $e = xy$ is an edge and there is $v_i \notin \{x, y\}$ such that $x \sim v_i$ and $y \not\sim v_i$ then e is good for w after the algorithm is finished. If, for an edge $e = xy$, there is no such vertex v_i, then it implies that $N[x] = N[y]$ and e is good from the beginning. $\square$

Comment: The upper bound could also be proved by induction on the number of edges by removing two incident edges whose endpoints do not induce a complete graph.

2.2 Proof of Theorem 3

Let $G=(V,E)$ be a graph of maximum degree Δ, let k be the coloring number of G and let $L : V \to 2^{\mathbb{R}}$ be a list assignment with $|L(v)| \geq (k-1)\cdot\Delta+1$, $v \in V$. Analogously to the proof of Theorem 2 we consider a coloring algorithm. Every vertex gets the initial label $w(v)=0$ regardless of $0 \in L(v)$. Analogously as above, let $xy \in E(G)$ be called bad for w if $S(x)=S(y)$, otherwise xy is good.

Consider an ordering $v_1,\ldots,v_n$ of the vertices of G such that for every i, the vertex v_i has degree at most $k-1$ in the subgraph G_i induced by $v_1,\ldots,v_i$. Note that by the definition of the coloring number, the minimum degree of any induced subgraph of G is at most $k-1$.

Step 1. Choose $w(v_1) \in L(v_1)$.

Step i, $2 \leq i \leq n$. Assume that the vertices $v_1,\ldots,v_{i-1}$ are labeled from their lists. We shall choose $w(v_i)$. Here, we consider two sets of edges in G_i: $E_1(v_i)$ is the set of edges in G_i incident to v_i and $E_2(v_i)$ consists of edges from G_i not incident to v_i and having exactly one endpoint adjacent to v_i. If $e \notin E_1(v_i)\cup E_2(v_i)$, then $w(v_i)$ contributes equally to the sums of both end vertices of e or to none of them.

For every good edge e in $E_1(v_i)\cup E_2(v_i)$ there is exactly one value w_e such that choosing $w(v_i)=w_e$ makes e bad. If there is a bad edge e in $E_1(v_i)\cup E_2(v_i)$, then it remains bad if and only if $w(v_i)=w_e=0$. Note that $|E_1(v_i)\cup E_2(v_i)| \leq k-1+(k-1)\cdot(\Delta-1)$. Since $L(v) \geq (k-1)\cdot\Delta+1$, we can choose a label $w(v_i) \in L(v_i)$ different from w_e for all $e \in E_1(v_i)\cup E_2(v_i)$. All edges in $E_1(v_i)\cup E_2(v_i)$ are good afterwards.

Consider an arbitrary edge $e=v_iv_j$ with $i<j$. Clearly, e belongs to all subgraphs G_t with $t \geq j$. At the jth step v_j gets its label and e becomes or remains good since $e \in E_1(v_j)$. In the following steps $t \in \{j+1,\ldots,n\}$ the edge e remains good since a good edge belonging to $E(G_t)$ cannot change its status. □

Comment: The idea above cannot be applied to find a constant bound for planar graphs. For every integer x there are planar graphs such that for any good ordering $v_1,\ldots,v_n$ as above and every sufficiently large step i, $|E_1(v_i)\cup E_2(v_i)| \geq x$. Consider for example $G=K_{2,x}$.

2.3 Proof of Theorem 5

First, we assume that G is a connected bipartite graph with partition sets A and B which is not a star. So, the parameter $\delta_{X,2}$ is well defined in that case, and it is at least 2. Let us assign the same label $w(x)=p$ to all vertices $x \in A$ and the same label $w(x)=q$ to all vertices $x \in B$. For two adjacent vertices $x \in A, y \in B$ with degrees d and d', resp.,

$$\sum_{v\in N[x]} w(v) - \sum_{v\in N[y]} w(v) = (p+qd)-(q+pd') = p(1-d')+q(d-1). \tag{1}$$

Let $Q=\{(d'-1)/(d-1) : d'=d(x), d=d(y), x \in A, y \in B, xy \in E(G)\}$. If there is a pair of relatively prime integers p,q such that $p/q \notin Q$ then the above coloring is good for closed neighborhoods. Now, observe that there are at least $c'm^2$ pairs of relatively prime numbers each of which at most m (where c' is some constant), see [18]. Since $|Q| \leq |E(G)|$, we can conclude that if $c'm^2 > |E(G)| \geq |Q|$, or equivalently $m > c\sqrt{|E(G)|}$, c - a constant, then this good coloring has labels each at most m.

Finally, let $p = \lfloor \frac{\Delta_A - 1}{\delta_{B,2}-1} \rfloor + 1$ and $q = 1$ (where the symmetric case can be proven analogously). Let $x \in A$ and $y \in B$ be adjacent vertices with degrees d, d', respectively. If $d' \geq 2$, then, using (1) we have that

$$\sum_{v \in N[x]} w(v) - \sum_{v \in N[y]} w(v) = p(1-d') + (d-1) \leq \left(\left\lfloor \frac{\Delta_A - 1}{\delta_{B,2} - 1} \right\rfloor + 1 \right) (1 - \delta_{B,2}) + (\Delta_A - 1) < 0.$$

If $d' = 1$ then $\sum_{v \in N[x]} w(v) - \sum_{v \in N[y]} w(v) \neq 0$ (unless $G = K_2$).

Now, consider a cycle C_n of length $n \geq 4$. Call two vertices coupled if they are the endpoints of a length 3 path. The labeling of a cycle is closed distinguishing if and only if any two coupled vertices are labeled differently. Construct a graph G' with $V(G') = V(C_n)$ and $uv \in E(G')$ if and only if u and v are coupled. Clearly, $\chi_\ell(G') = \mathrm{dis}_\ell[G]$ if G is a cycle. Moreover, it is well-known that $\chi_\ell(C_n) = 3$ if n is odd and $\chi_\ell(C_n) = 2$ if n is even (see [11]).

If G is an odd cycle then G' consists of one or more odd cycles, and thus $\chi_\ell(G') = \chi(G) = 3$. Analogously, if C_n is an even cycle and $n \neq 6$ then G' consists of one or more even cycles, and thus $\chi_\ell(G') = \chi(G) = 2$. Finally, G' is the union of three vertex disjoint K_2's if $n = 6$ and the same equality holds.

Therefore, $\mathrm{dis}_\ell[G] = \chi_\ell(G') = \chi(G)$ in all cases.

2.4 Proof of Theorem 6

To prove the first two items of the theorem, assume that T is a tree of order ≥ 3 and fix a root r of T of degree at least 2. Let $L(v)$ and $L^*(v)$, $v \in V(T)$ be lists of size 3, and of size 2, respectively, where the elements of $L^*(v)$ are positive, $v \in V(T)$.

Suppose that there is no closed distinguishing labeling of T from L or from L^*. For a labeling, we say as before that an edge xy is bad if $S[x] = S[y]$, otherwise an edge is good. Let $L'(v) = L(v) - \{0\}$ if v has degree ≥ 2, and $L'(v) = L(v)$ for the leaves v of T. For a labeling w, denote $\mathrm{lev}(w)$ the smallest depth of a bad edge under w. Pick assignments w, w^* of labels from L', L^* respectively, with the largest $\mathrm{lev}(w)$, $\mathrm{lev}(w^*)$. Let uv be a bad edge of w of the smallest depth, where v is further from the root r than u. Similarly define $u'v'$ for w^*.

Case 1 If v (v') is not a leaf, there is its neighbor z that is further from the root than v (v'). Change the label of z so that uv ($u'v'$) is good. Note that such switch does not affect the edges with depth smaller or equal than the depth of uv.

Case 2 If v is a leaf, let x be an ancestor of u. We have that v cannot be the only descendant of u, since otherwise $S[u] - S[v] = w(x) \neq 0$. Let $v_0 = v, v_1, \ldots, v_{k-1}$, $k \geq 2$, denote all descendants of u. Let $E = \{v_0u, \ldots, v_{k-1}u, ux\}$. We shall modify the labels of $v_0, \ldots, v_{k-1}$ so that all of the edges in E become good. First set the label of every v_i as

$\min L'(v_i)$, $i = 0, \ldots, k-1$. Then we repeat the following procedure until all the mentioned edges are good (index addition is modulo k). Pick any bad edge $e \in E$ and if $e = uv_i$ for some $i \in \{0, \ldots, k-1\}$, we choose the new label for v_{i+1} by picking the next larger value from its list, while if $e = ux$, we choose the new label for v_0 by picking the next larger value from its list. Note that if an edge $e \in E$ changed its status from bad to good, it could not become bad again since at each step $S[x]$ remains the same, $S[u]$ increases, $S[v_i]$ either increases by the same amount as $S[u]$ or remains the same. Thus, the label of v_i was changed at most once for $i = 1, \ldots, k-1$, and at most twice for $v_0 = v$. Since $|L'(v_i)| \geq 2$, $|L'(v)| = 3$, all the changes were feasible. As a result, all edges in E become good and edges with depth smaller or equal than the depth of uv are not affected. Performing such switches for all bad edges of the same depth as depth of uv yields a labeling where all bad edges have larger depth, a contradiction.

Case 3 Finally, if v' is a leaf, we get a contradiction since $S[u'] - S[v']$ is the sum of labels of other neighbors of u', that are positive.

Therefore, there is a closed distinguishing labeling of T from L and from L^*. The result for L^* implies that $\mathrm{dis}[T] \leq 2$.

To prove the last item of the theorem we use induction on the number of vertices of T with the basis consisting of all stars on at least 3 vertices.

Let T be a star with leaves $x_1, \ldots, x_k$, and a center x, $k \geq 2$. If $k \equiv 1 \pmod p$ then let $w(x) = 1$, $w(x_1) = w(x_2) = 2$, $w(x_i) = 1$, $i = 3, \ldots, k$. Then $S[x] = k + 3 \equiv 4 \pmod p$, $S[x_1] = S[x_2] = 3$, $S[x_i] = 2$, $i > 2$. If $k \not\equiv 1 \pmod p$ then give label 1 to all vertices of the star. Then $S[x] = k + 1 \not\equiv 2 \pmod p$, $w(x_i) = 2$, $i = 1, \ldots, k$. In each of these labelings $S[x]$ differs from $S[x_i]$ modulo p for all $i = 1, \ldots, k$.

For an induction step, consider a tree T of diameter at least 3 and a leaf x_1 adjacent to a vertex y such that y has exactly one neighbor, z, of degree greater than 1. Let $x_2, \ldots, x_k$ be all other leaves adjacent to y, if any exists. Let $T' = T - \{x_1, \ldots, x_k\}$ and w' be its labeling satisfying the conditions of the lemma. We shall extend it to the labeling w of T. Set $w(u) = w'(u)$ for all vertices $u \in V \setminus \{x_1, ..., x_k\}$. Note that then $S[v] = \sum_{u \in N[v]} w'(u) = \sum_{u \in N[v]} w(u)$ for $v \in V \setminus \{x_1, ..., x_k, y\}$.

If $k = 1$, then $S[x_1] \not\equiv S[y] \pmod p$ regardless of $w(x_1)$. Choose $w(x_1)$ so that $S[y] \not\equiv S[z] \pmod p$. We have that $S[y] = w(y) + w(z) + w(x_1)$ and $S[z]$ does not depend on $w(x_1)$. So, it is always possible to choose an appropriate value for $w(x_1)$.

If $k \geq 2$, we will extend the labeling by setting $w(x_1) = \alpha$ and $w(x_i) = \beta$ for $i = 2, ..., k$, with $\alpha, \beta \in \{1, 2, 3\}$. Then, $w(x_1) + ... + w(x_k) \equiv \alpha + (k-1)\beta \pmod p$. If

$$\begin{cases} \alpha + (k-1)\beta \not\equiv S[z] - w'(y) - w'(z) \pmod p, \\ (k-1)\beta + w'(z) \not\equiv 0 \pmod p, \quad \text{and} \\ \alpha + (k-2)\beta + w'(z) \not\equiv 0 \pmod p, \end{cases}$$

then yz, x_1y, and x_iy for $i = 2, ..., k$ comply with the requirements of the theorem. Thus first we choose $\beta \in \{1,2,3\}$ so that the second inequality is fulfilled. Such β always exists since $w'(z) \neq 0$. Next, choose $\alpha \in \{1,2,3\}$ satisfying the first and the third inequalities. This also can be done because we need to avoid at most two specific values for α.

2.5 Proof of Corollary 7

Let $F_1, \ldots, F_k$ be the strong induced forests covering $E(T)$. Color the vertices of each F_i according to the last item of Theorem 6 with colorings w_i, $i = 1, \ldots, k$, so that the role of p is played by p_i for each w_i, where p_is are pairwise co-prime numbers at least 4 each. Assign each vertex v of G a k-tuple $(w_1(v),\ w_2(v), \ldots,\ w_k(v)) \in \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_k}$. If a vertex v does not belong to a forest F_i then let $w_i(v) = 0$. If $uv \in E(G)$ then uv belongs to at least one forest F_i and thus the sums of the k-tuples over the closed neighborhoods of u and v are different in $\mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_k}$.

Finally, we use the same idea as Norin's. Since the group $\mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_k}$ is isomorphic to $\mathbb{Z}_n$, where $n = p_1 p_2 \ldots p_3$, consider a corresponding isomorphism ϕ and assign each vertex v a label $\phi((w_1(v), \ldots, w_k(v)))$. Note also that none of these labels is equal 0.

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