ON MULTIPLE COVERINGS OF THE INFINITE RECTANGULAR GRID WITH BALLS OF CONSTANT RADIUS

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ABSTRACT. We consider the coverings of graphs with balls of constant radius satisfying special multiplicity condition. A (t, i, j)-cover of a graph $G = (V, E)$ is a subset S of vertices, such that every element of S belongs to exactly i balls of radius t centered at elements of S and every element of $V \setminus S$ belongs to exactly j balls of radius t centered at elements of S. For the infinite rectangular grid, we show that in any (t, i, j)-cover, i and j differ by at most $t + 2$ except for one degenerate case. Furthermore, for i and j satisfying $|i - j| > 4$ we show that all (t, i, j)-covers are the unions of the diagonals periodically located in the grid. Also, we give the description of all $(1, i, j)$-covers.

1. INTRODUCTION

In this paper we address the problem of covering the infinite rectangular lattice with balls of constant radius such that the covering satisfy special multiplicity conditions. Let $G = (V, E)$ be a graph, and choose $v \in V$. A t-neighborhood of v, $N_t(v)$, is a ball with radius t centered at x. An open t-neighborhood of v is $N_t(v) \setminus v$.

Definition 1. The set S of vertices is called a (t, i, j)-***cover*** or (t, i, j)-***covering code*** if every element of S belongs to exactly i balls of radius t centered at elements of S and every element of $V \setminus S$ belongs to exactly j balls of radius t centered at elements of S. We shall frequently call the balls of radius t centered about elements of S ***covering balls***.

The notion of (t, i, j)-covering generalizes the notion of domination and perfect codes in graphs, which arised as a generalization of classical notions of Hamming- and Lee-error-correcting codes [1].

For instance, the classical t-perfect code in graphs corresponds to $(t, 1, 1)$-covering code, i.e., the subset of vertices $S \subseteq V$ such that every vertex is at distance at most t from exactly one element of S. It was shown by Kratochvil [7] that the problem of finding a t-perfect dominating set in graphs (a $(t, 1, 1)$-cover) is NP-complete and that almost every graph does not have a perfect dominating set. Moreover, he showed that this problem is NP-complete even when restricted only to bipartite graphs with maximum degree three. For more results, see [8], [9], [2].

On the other hand, $(1, i, j)$-covering code is exactly $[j, i]$-domination set defined by Telle [11]. We say that the subset S of vertices is $[j, i]$-dominating set iff $|S \cap N_1(x)| = i$ for all $x \in S$ and $|S \cap N_1(x)| = j$ for all $x \notin S$. Telle proved that the problem of deciding whether a graph has $[j,i]$-dominating set is NP-complete even in the case of a planar bipartite graph of maximum degree three.

One of the motivation for studying covering codes in special graphs is a network communication problem. As an example of $(t,1,2)$-cover, consider a system of transmitter stations for a cellular phone network. We locate transmitter stations in the vertices of the corresponding graph such that each station is not within a reaching distance t from any other station (to avoid interference), but any other vertex is reached from exactly two transmitter stations. Surprisingly, (t,i,j)-coverings do not exist for every value of i and j even in the case of a graph as simple as the infinite rectangular grid. For example, $(1,1,3)$-cover of the infinite rectangular grid does not exist as shown in section 3.

In this paper, we give criteria on parameters t, i and j for (t,i,j)-covers of infinite rectangular grid to exist and describe explicitly most of them by splitting all (t,i,j)-covers into two classes. We treat these classes separately in sections 4 and 5. We show that either i and j do not differ much or (t,i,j)-covers have a simple periodic structure. For $t>1$ we prove that $|i-j|\le t+2$ unless $|i-j|=2t+1$, in which case the covering is unique. Furthermore, for $|i-j|\ge 5$ we explicitly describe all possible (t,i,j)-covers. We describe all $(1,i,j)$-covers in section 3. Before stating and proving the results, we reformulate the covering problem in terms of special vertex-coloring of the infinite rectangular grid in the next section.

2. Isotropic colorings and (t,i,j)-covers

The problem of finding a (t,i,j)-cover of a graph G is equivalent to the following vertex-coloring problem of G. Color the vertices of G in two colors, black and white, such that black vertices correspond to the centers of the covering balls. Thus, the t-neighborhood of every black vertex contains exactly i black vertices and the t-neighborhood of every white vertex contains exactly j black vertices. A similar coloring problem was introduced by Vizing [12], [13]. He considered a ***distributive*** or ***isotropic*** coloring of a graph, that is, a coloring in which the number of vertices of a fixed color in the 1-neighborhood of any vertex depends only on the color of this vertex. We shall state and prove covering results in terms of colorings.

Let $\varphi : V(L) \to \{0,1\}$ be a coloring of $V(L)$. We call a vertex $\mathbf{u}\in V(L)$ ***black*** if $\varphi(\mathbf{u})=1$ and we call a vertex $\mathbf{u}$ ***white*** if $\varphi(\mathbf{u})=0$. For $\mathbf{u}\in V(L)$ and $k\in\{0,1\}$, let $N_t^k(\mathbf{u})$ be the set of vertices of color k in the t-neighborhood of $\mathbf{u}$.

Definition 2. A coloring φ of L is (t,i,j)-***isotropic*** if every black vertex has exactly i black vertices within distance t and every white vertex has exactly j black vertices within distance t.

Remark 3. $\varphi : V(L)\to\{0,1\}$ is a (t,i,j)-isotropic coloring iff $S=\{\mathbf{v}\in V(L):\ \varphi(\mathbf{v})=1\}$ is a (t,i,j)-cover of L.

Further, we shall investigate (t,i,j)-isotropic colorings. We shall use the following definitions and notations throughout the paper. Often, we denote the vertices of L by ordered pairs

(x- and y-coordinates taking integer values). The set

$$L_e = \{(x, y) \in V(L) : \ x + y = 0 \ (mod\ 2)\}$$

is called ***even*** sub-lattice of L and

$$L_o = \{(x, y) \in V(L) : \ x + y = 1 \ (mod\ 2)\}$$

is called ***odd*** sub-lattice of L. Further, by ***sub-lattice*** we mean either L_e or L_o. We call a set $\{(x, x + c) : x \in \mathbb{Z}\}$ a ***right diagonal*** and $\{(x, -x + c) : x \in \mathbb{Z}\}$ a ***left diagonal***. If all elements of a subset $S \subseteq V(L)$ under coloring φ have color 1, then we write $\varphi(S) = 1$ and call S ***black***. Similarly, we call S ***white*** when φ is identically 0 on S, we write $\varphi(S) = 0$ in this case. In general, if all elements of S have the same color, we call S ***monochromatic***.

Definition 4. We call a coloring of $V(L)$ a ***diagonal coloring*** if even and odd sub-lattices are the disjoint unions of monochromatic diagonals.

Definition 5. We call $\{0, 1\}$-coloring φ of a line $X_a = \{(x, a) : x \in \mathbb{Z}\}$, a t-***periodic*** coloring if $\varphi((x, a)) = \varphi((x + t, a))$ for all $(x, a) \in X_a$. We call such a coloring t-***anti-periodic*** if $\varphi((x, a)) = 1 - \varphi((x + t, a))$ for all $(x, a) \in X_a$. Similarly, we call a diagonal coloring of $V(L)$ t-***periodic*** or t-***anti-periodic*** if horizontal lines of L are colored t-periodically or t-anti-periodically.

For $t = 1$, we describe all such colorings in the next section. For $t > 1$, we partition all (t, i, j)-isotropic colorings of L into two classes, colorings of Type A and colorings of Type B according to the way of coloring the neighborhoods $N(x, y)$ of the points in the grid.

A 2-coloring of the infinite rectangular grid is ***Type A*** if some vertex has odd number of neighbors of each color or has its horizontal neighbors in one color and its vertical neighbors in other color.

A 2-coloring of the infinite rectangular grid is ***Type B*** if every vertex $\mathbf{v}$ has all its neighbors of the same color or has two neighbors of each color that do not lie on a line through $\mathbf{v}$. We study colorings of Type A and Type B in sections 4 and 5 respectively.

First we notice that for any (t, i, j)-isotropic colorings i and j do not differ by more than $2t + 1$.

Lemma 6. *For all (t, i, j)-isotropic colorings, $|i - j| \leq 2t + 1$.*

Proof. Consider two points (x, y) and $(x + 1, y)$ such that $\varphi((x, y)) = 0$ and $\varphi((x + 1, y)) = 1$. Such two points exist because otherwise we have the monochromatic coloring of L and the bound is trivial in this case. Now notice that $|N_t(x, y) \setminus N_t(x + 1, y)| = 2t + 1$. Therefore $|i - j| \leq 2t + 1$ as well. In the case when L_e is black and L_o is white, i.e., when the coloring is 2-periodic and not monochromatic, we have $|i - j| = 2t + 1$. $\square$

3. Description of all $(1,i,j)$-colorings

In this section we describe all $(1,i,j)$-colorings. We often call $(1,i,j)$-colorings simply ***isotropic*** colorings. An isotropic coloring $\varphi : V(L) \to \{0,1\}$ corresponds to a matrix $A = \{a_\varphi(k,l) : k,l \in \{0,1\}\}$, where $a_\varphi(k,l)$ is the number of vertices of color l in the open neighborhood of a vertex of color k. We call this matrix A a ***coloring matrix*** corresponding to φ. It is clear that for any coloring matrix

$$a_\varphi(k,l) \in \{0,\dots,4\} \tag{1}$$

for $k,l \in \{0,1\}$ and

$$a_\varphi(k,0) + a_\varphi(k,1) = 4 \tag{2}$$

for $k \in \{0,1\}$, unless $a_\varphi(0,0) = 4$ or $a_\varphi(1,1) = 4$ in which case the coloring is monochromatic. Therefore, to describe all $(1,i,j)$-colorings we consider all 2×2 matrices satisfying (1) and (2).

We say that two colorings $\varphi_1 : V(L) \to \{0,1\}$ and $\varphi_2 : V(L) \to \{0,1\}$ are equivalent (and the corresponding coloring matrices are equivalent) if one is obtained from another by switching colors. We shall consider only one coloring from each equivalence class.

Clearly, if φ_1 and φ_2 are equivalent then $a_{\varphi_1}(k,l) = a_{\varphi_2}(1-k,1-l)$ for all $k,l \in \{0,1\}$. Also, notice that if $a_\varphi(1,0) = 0$ or $a_\varphi(0,1) = 0$ then the coloring is monochromatic, using only black or only white, respectively. Thus, we need to consider only the following coloring matrices:

$$A_1 = \begin{pmatrix} 0 & 4 \\ 1 & 3 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & 4 \\ 2 & 2 \end{pmatrix} \quad A_3 = \begin{pmatrix} 0 & 4 \\ 3 & 1 \end{pmatrix} \quad A_4 = \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}$$

$$A_5 = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix} \quad A_6 = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \quad A_7 = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \quad A_8 = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$$

$$A_9 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \quad A_{10} = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}.$$

We show first that the matrix A_3 does not correspond to any isotropic coloring. Consider a white vertex $\mathbf{u}$ and its neighbor $\mathbf{v}$. Since all neighbors of $\mathbf{u}$ are black, we have only two non-isomorphic ways to color neighbors of $\mathbf{v}$. As shown on Figure 1, in each case there is a black vertex $\mathbf{x}$ which has two black neighbors, which is impossible.

Theorem 7. *A 2×2 matrix is the coloring matrix for an isotropic coloring iff it is one of or is equivalent to one of the following: A_1, A_2, A_4, A_5, A_6, A_7, A_8, A_9, A_{10}. Matrices A_4 and A_9 correspond to uncountably many isotropic colorings. The other matrices in the list correspond to the unique (up to isomorphism) isotropic colorings.*

FIGURE 1. The initial configurations determined by A_3

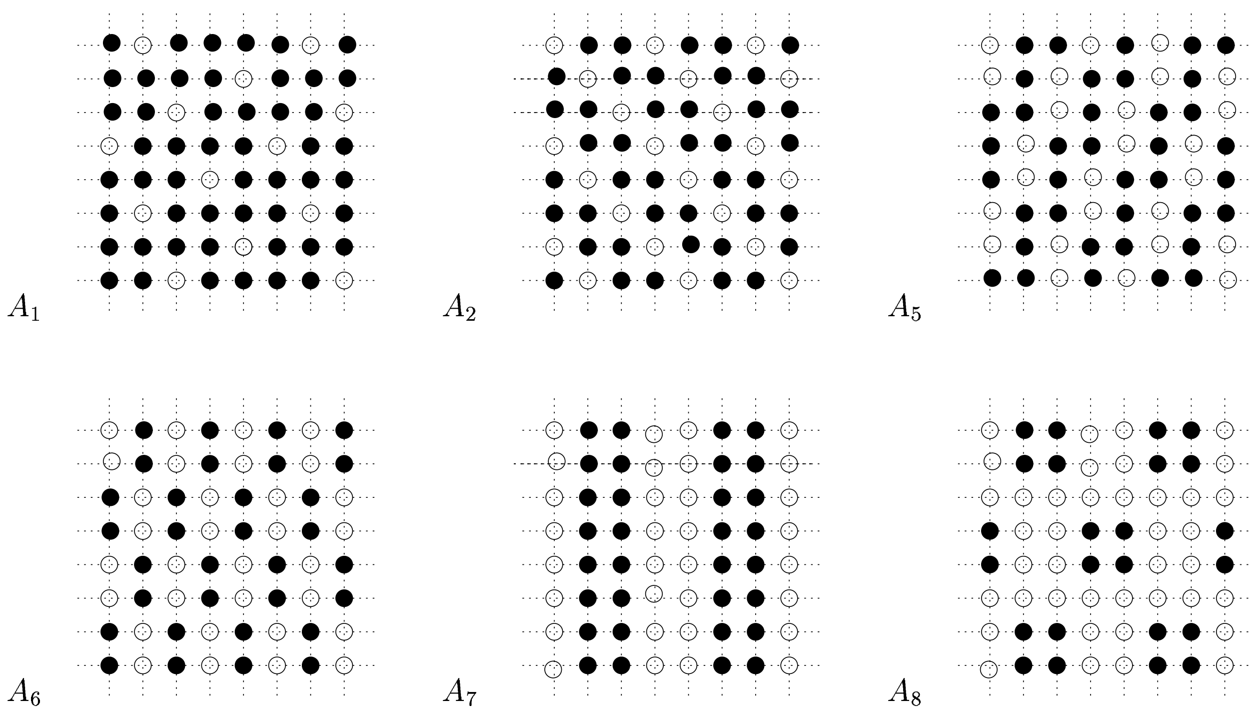

FIGURE 2. Colorings corresponding to matrices A_1, A_2, A_5, A_6, A_7 and A_8.

Proof. To show that the matrices $A_1,\ A_2,\ A_5,\ A_6,\ A_7,\ A_8,\ A_{10}$ correspond to unique (up to isomorphism) isotropic colorings it suffices to notice (see Figure 2) that these colorings are unambiguously determined (up to isomorphism) by the initial configurations consisting of a point and its neighborhood. The matrix A_{10} corresponds to the monochromatic coloring.

Next we describe colorings corresponding to A_9 and A_4. It is clear that φ is an isotropic coloring corresponding to A_9 if and only if every vertex has exactly two black neighbors. Similarly, φ is an isotropic coloring corresponding to A_4 if and only if every vertex has exactly three black neighbors. Call these restrictions the ***color conditions*** for A_9 and A_4, respectively. Since the neighborhood $N(\mathbf{u})$ of any point $\mathbf{u}$ is a subset of L_e or L_o, we may color L_o and L_e separately. We say that the coloring of a sub-lattice L_o or L_e is ***good*** if it satisfies the color condition for A_9 (respectively A_4) for every points on the other sub-lattice.

Claim 1. A_9 corresponds to uncountably many isotropic colorings.
If, in a good coloring, a diagonal D has two consecutive vertices of the same color, then we claim that the coloring of D determines the coloring of the sub-lattice containing it. Let D be the right diagonal containing (x,y). If $\varphi((x,y)) = \varphi((x+1,y+1)) = 0$, then $\varphi((x+2,y)) = \varphi((x+1,y-1)) = 1$. Next $\varphi((x+3,y+1))$ and $\varphi((x,y-2))$ are determined by $\varphi((x+2,y+2))$ and $\varphi((x-1,y-1))$. Thus the coloring of D determines the coloring of the diagonal next to D in each direction. Since the diagonal next to D also has two consecutive vertices of the same color, the argument propagates to determine the coloring of the whole sub-lattice.

Suppose that there is no diagonal with two consecutive vertices of the same color. We call such a coloring of a sub-lattice *alternated*. Thus every good coloring of L_e is either an alternated coloring or is determined by the coloring of one of the diagonals which has two consecutive vertices of the same color. Therefore the number of good colorings of L_e, and, correspondingly of L_o, is equal to the number of doubly infinite binary sequences. Thus there are uncountably many such colorings and Claim 1 follows.

Claim 2. A_4 corresponds to uncountably many isotropic colorings.
We say that the coloring of the diagonal is *odd* if the number of black vertices between two successive white vertices on this diagonal is always odd. In particular, there are no two consecutive white vertices in the odd coloring of a diagonal.

We claim that if the coloring corresponding to matrix A_4 is good then every diagonal is odd-colored. Suppose, without loss of generality, that there are two successive white vertices with even number of black vertices between then on the diagonal. That is, $\varphi((x,y)) = \varphi((x+2i+1, y+2i+1)) = 0$ and $\varphi((x+q,y+q)) = 1$ for all $1 \leq q \leq 2i$. Then either $\varphi((x+2+2q, y+2q)) = 1$ and $\varphi((x+2+(2q+1), y+(2q+1)) = 0$ for all $0 \leq q \leq i-1$ or $\varphi((x+2+2q, y+2q)) = 0$ and $\varphi((x+2+(2q+1), y+(2q+1)) = 1$ for all $0 \leq q \leq i-1$. In the first case, we have that the point $(x+2i+1, y+2i)$ has at least two white neighbors, in the second case, the same holds for the point $(x+1,y)$, a contradiction.

Note that any odd-coloring of a diagonal extends to a unique good coloring of its sub-lattice by determining the coloring of the diagonals next to a diagonal with already known coloring. This is done as in case of A_9. Namely, if $\varphi((x,y)) = 0$ and $\varphi((x+1,y+1)) = 1$ then $\varphi((x+1,y-1)) = \varphi((x+2,y)) = 1$. Next, $\varphi((x,y-2))$ and $\varphi((x+3,y+1))$ are determined by $\varphi((x-1,y-1))$ and $\varphi((x+2,y+2))$. In case when there is a black monochromatic diagonal, the diagonal next to it has an alternating coloring, the next is monochromatic black, etc.

Therefore, as in case of A_9, the coloring of one diagonal determines (up to isomorphism) the good coloring of the sub-lattice it belongs to. Thus we have established a bijection between the set of isotropic colorings with coloring matrix A_4 and the set of doubly infinite sequences of odd numbers. Thus the set of isotropic colorings is uncountable and Claim 2 follows.

This concludes the proof of Theorem 7 and the description of all $(1,i,j)$-isotropic colorings.

□

4. Colorings of Type A

In this section we consider (t, i, j)-isotropic colorings of Type A for arbitrary $t > 1$. Namely, we study the (t, i, j)-isotropic colorings such that some vertex has odd number of neighbors of each color or has its horizontal neighbors in one color and its vertical neighbors in other color. We show that in such colorings i and j do not differ by more than four.

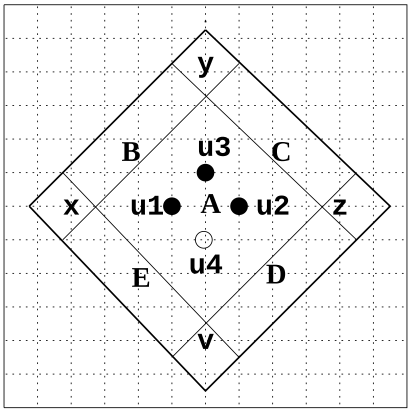

Figure 3. Type A coloring

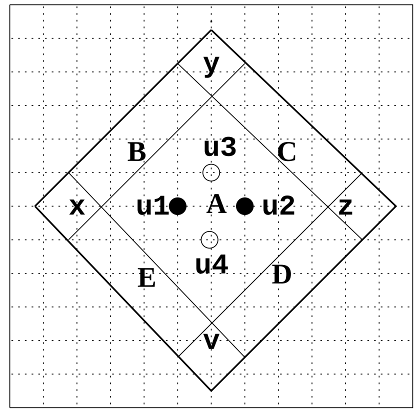

Figure 4. Type A coloring

Theorem 8. *If φ is a (t, i, j)-isotropic coloring of type A then $|i - j| \leq 4$.*

Proof. For an arbitrary point $(p, q) \in V(L)$, consider its open neighborhood $\mathbf{u}_1 = (p-1, q)$, $\mathbf{u}_2 = (p+1, q)$, $\mathbf{u}_3 = (p, q+1)$, $\mathbf{u}_4 = (p, q-1)$. We denote by A, B, C, D, E, x, y, z and v the number of the black points in the corresponding subsets of $N_{t+1}((p, q))$ as follows (see Figure 3).

$$A = |N_t^1(\mathbf{u}_1) \cap N_t^1(\mathbf{u}_2) \cap N_t^1(\mathbf{u}_3) \cap N_t^1(\mathbf{u}_4)|,$$

$$\begin{aligned} B &= |N_t^1(\mathbf{u}_1) \cap N_t^1(\mathbf{u}_3)| - A, & x &= |N_t^1(\mathbf{u}_1)| - A - B - E, \\ D &= |N_t^1(\mathbf{u}_2) \cap N_t^1(\mathbf{u}_4)| - A, & z &= |N_t^1(\mathbf{u}_2)| - A - C - D, \\ C &= |N_t^1(\mathbf{u}_3) \cap N_t^1(\mathbf{u}_2)| - A, & y &= |N_t^1(\mathbf{u}_3)| - A - B - C, \\ E &= |N_t^1(\mathbf{u}_1) \cap N_t^1(\mathbf{u}_4)| - A, & v &= |N_t^1(\mathbf{u}_4)| - A - D - E. \end{aligned}$$

Notice that x, y, z and v are at most 2. As shown in the figure, the first case gives us the following by considering the number of black points in the t-neighborhoods of $\mathbf{u_3}$, $\mathbf{u_2}$, $\mathbf{u_4}$ and $\mathbf{u_1}$, respectively.

$$A + B + C + y = i, \quad A + C + D + z = i, \quad A + D + E + v = j, \quad A + E + B + x = i.$$

By subtracting the second equation from the third and the first from the forth, we obtain $j - i = E - C + v - z$, $\quad 0 = i - i = E - C + x - y$. Combining these two equations gives us $j - i = y - x + v - z$, thus $|j - i| \leq 4$. We get the same result in case when (p, q) has three white neighbors.

In the second case shown in the figure we have the following number of black points in the t-neighborhoods of $\mathbf{u_3}$, $\mathbf{u_2}$, $\mathbf{u_4}$ and $\mathbf{u_1}$, respectively.

$$A+B+C+y=j, \quad A+C+D+z=i, \quad A+D+E+v=j, \quad A+E+B+x=i.$$

By subtracting the second equation from the first we obtain $j-i=B-D+y-z$. On the other hand, as we subtract the forth equation from the second and third from the first, we have the following $0=i-i=C+D-B-E+z-x=j-j=B+C-D-E+y-v$. Thus $D-B=(y-v+x-z)/2$ and $j-i=(v-y-x+z)/2+y-z=(v+y-x-z)/2$. Therefore $|j-i|\leq 2\leq 4$. □

5. Colorings of Type B

In this section we describe all (t,i,j)-isotropic colorings of Type B for $t>1$. Namely, we study the colorings such that every vertex $\mathbf{v}$ has all its neighbors of the same color or has two neighbors of each color that do not lie on a line through $\mathbf{v}$. We show that all such colorings have a special periodic structure, moreover, the infinite rectangular grid under such coloring is the union of monochromatic diagonals.

Proposition 9. *If φ is a (t,i,j)-isotropic coloring of type B, then it is a diagonal coloring.*

Proof. Suppose that a coloring of type B restricted to a sub-lattice is not diagonal. Then, without loss of generality, there is a right diagonal D which is not monochromatic, i.e., one having two successive vertices of different color. By looking at these vertices and the vertices next to them on the parallel diagonal in this sub-lattice, it is easy to see that the coloring of D can be uniquely extended to the coloring of the diagonal next to it and, further, to the coloring of the whole sub-lattice. In this case all left diagonals are monochromatic. □

Thus it follows that coloring of type B is determined by the coloring of any horizontal line $\{(x,a) : x\in\mathbb{Z}\}$ and by the orientation of the monochromatic diagonals in L_e and L_o. We shall prove that there are only few possible colorings of a horizontal line determining a (t,i,j)-isotropic coloring of L. In particular, these colorings must be periodic. Next we show that a coloring is of Type B iff it is one of the following diagonal colorings for $p=t$ or $p=t+1$.

Coloring 1. p-periodic coloring, p is odd and the monochromatic diagonals of even sub-lattice and odd sub-lattice are parallel.

Coloring 2. p-anti-periodic coloring, p is even.

Coloring 3. p-periodic coloring, p is even and for every horizontal or vertical interval I of length p the number of black points from the even sub-lattice and from the odd sub-lattice is the same.

Coloring 4. 2 or 3-periodic coloring.

Theorem 10. *A coloring of type B is (t, i, j)-isotropic coloring iff it is one of the diagonal* **Colorings 1.–4.** *with $p = t$ or $p = t+1$. Furthermore, in a coloring of type B, $|i-j| \leq t+2$ unless the coloring is non-monochromatic and 2-periodic, in which case $|i-j| = 2t+1$.*

6. Proof of theorem 10

To prove Theorem 10 we shall describe explicitly all possible colorings of type B. As noted before, every coloring of Type B is determined by the coloring of any horizontal line and the orientation of the diagonals. The arrangement of the points in the grid yields two types of diagonals intersecting the t-neighborhood of (x, y) – the ***long diagonals*** whose intersection with $N_t((x, y))$ has size $t+1$, and the ***short diagonals*** whose intersection with $N_t((x, y))$ has size t. We define various notations to count long and short monochromatic diagonals of (x, y) intersecting specified intervals of length $2t+1$, $t+1$ and t in the horizontal line containing (x, y).

Interval and color of diagonal	$[(x-t, y), (x+t, y)]$ black	$[(x-t, y), (x-1, y)]$ color i	$[(x-t, y), (x, y)]$ color i
Number of long diagonals	$l((x, y))$	$l_i((x, y))$	$l'_i((x, y))$
Number of short diagonals	$k((x, y))$	$k_i((x, y))$	$k'_i((x, y))$

In our proof we shall frequently use the horizontal axis $H = \{(x, 0) : x \in \mathbb{Z}\}$ and will write $x = (x, 0)$ to be short. For the arbitrary vertices in the grid we use the bold letters. We use $l(x)$, $k(x)$, $l_i(x)$, $k_i(x)$, $l'_i(x)$ and $k'_i(x)$ for the respective values $l(x, 0)$, $k(x, 0)$, $l_i(x, 0)$, $k_i(x, 0)$, $l'_i(x, 0)$ and $k'_i(x, 0)$. The following lemmas give some properties of diagonal isotropic colorings and corresponding functions k and l.

Lemma 11. *If φ is a diagonal (t, i, j)-isotropic coloring and $\varphi(\mathbf{u}) = \varphi(\mathbf{v})$ then $k(\mathbf{u}) = k(\mathbf{v})$ and $l(\mathbf{u}) = l(\mathbf{v})$.*

Proof. The number of black points in the t-neighborhood of a vertex is equal to the number of points in black short and long diagonals. If $\varphi(\mathbf{u}) = \varphi(\mathbf{v}) = 1$, then the number of black points in the t-neighborhoods of both points is $i = tk(\mathbf{u}) + (t+1)l(\mathbf{u}) = tk(\mathbf{v}) + (t+1)l(\mathbf{v})$. Therefore $k(\mathbf{u}) - k(\mathbf{v}) = l(\mathbf{u}) - l(\mathbf{v}) + \frac{l(\mathbf{u})-l(\mathbf{v})}{t}$. Also $l(\mathbf{u}) \leq t+1$ for all $\mathbf{u}$, thus $l(\mathbf{u}) - l(\mathbf{v})$ is divisible by t iff one of the following cases holds:
a) $l(\mathbf{u}) = l(\mathbf{v})$,
b) $(l(\mathbf{u}) = t$ and $l(\mathbf{v}) = 0)$ or $(l(\mathbf{u}) = 0$ and $l(\mathbf{v}) = t)$,
c) $(l(\mathbf{u}) = t+1$ and $l(\mathbf{v}) = 1)$ or $(l(\mathbf{u}) = 1$ and $l(\mathbf{v}) = t+1)$.
In cases b) and c), we have that $|k(\mathbf{u}) - k(\mathbf{v})| = t+1$ which is impossible since $k(\mathbf{u}) \leq t$ for all

$\mathbf{u}$. The case a) gives us $l(\mathbf{u}) = l(\mathbf{v})$ and $k(\mathbf{u}) = k(\mathbf{v})$. Similarly, the same result holds when $\varphi(\mathbf{u}) = \varphi(\mathbf{v}) = 0$. □

Lemma 12. *If φ is a diagonal (t,i,j)-isotropic coloring and $\varphi(x) = \varphi(x+1)$ then $\varphi(x-t) = \varphi(x+1+t)$.*

Proof. Suppose that $\varphi(x) = \varphi(x+1)$, $\varphi(x-t) = 1$, and $\varphi(x+1+t) = 0$. Since all short diagonals of x correspond to long diagonals of $x+1$ and all long diagonals of x except the leftmost one correspond to short diagonals of $x+1$, we have $l(x) - 1 = k(x+1)$ and $l(x+1) = k(x)$. Therefore $l(x) - 1 = l(x+1)$. The Lemma 11 yields $l(x) = l(x+1)$, we have a contradiction. The case when $\varphi(x-t) = 0$ and $\varphi(x+1+t) = 1$ is similar. □

Lemma 13. *If φ is a diagonal t-anti-periodic (t,i,j)-isotropic coloring, then t is even. If φ is a diagonal $(t+1)$-anti-periodic (t,i,j)-isotropic coloring, then t is odd.*

Proof. Suppose φ is t-anti-periodic and t is odd. Let $\varphi(x) = 1$ and $\varphi(x+1) = 1$. By Lemma 11, $l(x) = l(x+1)$ and $k(x) = k(x+1)$. On the other hand, since φ is t-anti-periodic, then $\varphi(x-t) = \varphi(x+t) = 0$. Therefore, the short black diagonals of x correspond to the long black diagonals of $x+1$ and visa versa. That is $l(x+1) = k(x)$, $k(x+1) = l(x)$, thus $l(x) = k(x)$. Since φ is t-anti-periodic, $l(x) = l_1(x) + k_0(x)$ and $k(x) = k_1(x) + l_0(x)$. Thus $t = (k_1(x) + l_0(x)) + (k_0(x) + l_1(x)) = 2(k_1(x) + l_0(x))$, which is impossible for odd t. A similar argument holds for the $(t+1)$-anti-periodic coloring. □

Lemma 14. *Let φ be a diagonal (t,i,j)-isotropic coloring. If φ is t-periodic and t is odd, or if φ is $(t+1)$-periodic and t is even, then the monochromatic diagonals of even and odd sub-lattices are parallel.*

Proof. Suppose that L_e is the union of right monochromatic diagonals and that L_o is the union of left monochromatic diagonals.

Case 1. t is even and φ is $(t+1)$-periodic.

Consider a point $(x,0) \in L_e$. Thus $\varphi((x,0)) = \varphi((x+1,1))$. By comparing the t-neighborhoods of $(x,0)$ and $(x+1,1)$, we observe that $\varphi((x-(t-1),0)) = \varphi((x+(t+1),0))$. Similarly, if $(x,0) \in L_o$ then $\varphi((x,0)) = \varphi((x+1,-1))$, therefore $\varphi((x-(t-1),0)) = \varphi((x+(t+1),0))$ and the coloring of H is $(2t)$-periodic. Since φ is $(t+1)$-periodic as well, then φ is periodic with even and odd period which is possible only for monochromatic coloring. In this case the diagonals of even and odd sub-lattices are parallel.

Case 2. t is odd and φ is t-periodic.

If $(x,0) \in L_e$, then $\varphi((x,0)) = \varphi((x+1,1))$; therefore $\varphi((x-t,0)) = \varphi((x+t+2,0))$. If $(x,0) \in L_o$, then $\varphi((x,0)) = \varphi((x+1,-1))$; therefore $\varphi((x-t,0)) = \varphi((x+t+2,0))$, so the coloring of H is $(2t+2)$-periodic. Since φ is t-periodic then φ is periodic with even and odd period. Thus φ is monochromatic coloring with diagonals of odd and even sub-lattices being parallel. □

Lemma 15. *Let φ be a diagonal t-periodic (t,i,j)-isotropic coloring with t even. Then the number of black points from the even sub-lattice is the same as the number of black points from the odd sub-lattice in any horizontal or vertical interval I of length t.*

Proof. Without loss of generality, we may consider a horizontal interval, I, of length t on the horizontal axis. We shall omit the y-coordinate and write $(x,0)=x$ for all $x\in\mathbb{Z}$ and $I=[x-t,x-1]$ as before. There are four possible ways to color x and $x+1$. In each of these cases, we consider the number of short and long black diagonals intersecting $[x-t,x-1]$, that is $l_1(x)$ and $k_1(x)$. Since the long diagonals of x belong to the even sub-lattice and the short diagonals of x belong to the odd sub-lattice, or visa versa, then $l_1(x)$ corresponds to the number of black points in I from the even sub-lattice and $k_1(x)$ corresponds to the number of black points in I from the odd sub-lattice, or visa versa. We shall prove that $l_1(x)=k_1(x)$ for all x, thus proving the statement of the lemma.

We may assume that there are two consecutive vertices of the same color, otherwise the claim of the Lemma holds trivially. Assume that there is x such that $\varphi(x)=\varphi(x+1)=0$. Then, since the coloring is t-periodic, the number of black diagonals intersecting $[x-t,x+t]$ can be expressed through the number of black diagonals intersecting $[x-t,x-1]$, that is $l(x)=2l_1(x)$, $k(x)=2k_1(x)$ and $l(x+1)=2k_1(x)$. Since $l(x)=l(x+1)$, we have $l_1(x)=k_1(x)$. Moreover, $l(x)=k(x)$ for all x such that $\varphi(x)=0$.

If $\varphi(x)=0$ and $\varphi(x+1)=1$, then $l(x)=2l_1(x)$, $k(x)=2k_1(x)$, and $l(x+1)=2k_1(x)+1$. Since $l(x)=k(x)$, we have $k_1(x)=l_1(x)$. Moreover, $l(x)=l(y)-1$ for all x and y such that $\varphi(x)=0$ and $\varphi(y)=1$.

If $\varphi(x)=1$ and $\varphi(x+1)=0$, then $l(x)=2l_1(x)+1$ and $l(x+1)=2k_1(x)$. Since $l(x)=l(x+1)+1$, we have $k_1(x)=l_1(x)$.

Now if we have two consecutive vertices of color 1 as well, i.e., $\varphi(x)=1$ and $\varphi(x+1)=1$, then $l(x)=2l_1(x)+1$ and $l(x+1)=2k_1(x)+1$. Since $l(x)=l(x+1)$, we have $l_1(x)=k_1(x)$. □

Similar argument gives the following.

Lemma 16. *Let φ be a $(t+1)$-periodic diagonal (t,i,j)-coloring and t be odd. Then the number of black points from the even sub-lattice is the same as the number of black points from the odd sub-lattice in any horizontal or vertical interval I of length $t+1$.*

In the following Remark we list all possible colorings of vertices with horizontal coordinates x, $x+1$, $x-t$, and $x+t+1$ (the cases $(\varphi(x)=0,\ \varphi(x+1)=1)$ and $(\varphi(x)=1,\ \varphi(x+1)=0)$ are symmetric). We give the resulting relationship between the functions k and l at x and $x+1$.

Remark 17. Suppose that φ is a diagonal (t,i,j)-isotropic coloring. If $\varphi(x)=0$ and $\varphi(x+1)=1$, then we have the following cases:

A. If $\varphi(x-t)=1$ and $\varphi(x+1+t)=1$, then $k(x)=l(x+1)-1$ and $l(x)-1=k(x+1)$.

B. If $\varphi(x-t)=0$ and $\varphi(x+1+t)=1$, then $l(x+1)-1=k(x)$ and $l(x)=k(x+1)$.

C. If $\varphi(x-t)=0$ and $\varphi(x+1+t)=0$, then $l(x)=k(x+1)$ and $k(x)=l(x+1)$.
D. If $\varphi(x-t)=1$ and $\varphi(x+1+t)=0$, then $l(x)=k(x+1)+1$ and $l(x+1)=k(x)$.
If $\varphi(x)=\varphi(x+1)$, then we have the following cases:
1. If $\varphi(x)=\varphi(x+1)=1$ and $\varphi(x-t)=\varphi(x+1+t)=0$, then $l(x)=k(x)$.
2. If $\varphi(x)=\varphi(x+1)=0$ and $\varphi(x-t)=\varphi(x+1+t)=1$, then $l(x)-1=k(x)$.
3. If $\varphi(x)=\varphi(x+1)=1$ and $\varphi(x-t)=\varphi(x+1+t)=1$, then $l(x)-1=k(x)$.
4. If $\varphi(x)=\varphi(x+1)=0$ and $\varphi(x-t)=\varphi(x+1+t)=0$, then $l(x)=k(x)$.

If φ is a diagonal (t,i,j)-isotropic coloring, then for all x such that $\varphi(x)=0$ and $\varphi(x+1)=1$ only one of the cases A, B, C and D of Remark 17 holds. For all x such that $\varphi(x)=\varphi(x+1)=0$ only one of the cases 2 and 4 of Remark 17 holds. For all x such that $\varphi(x)=\varphi(x+1)=1$ only one of the cases 1 and 3 of Remark 17 holds, otherwise the conclusion of Lemma 11 does not hold.

proof of Theorem 10.
($\Longrightarrow$) First we show that if the coloring is (t,i,j)-isotropic of Type B, then it is one of **Colorings 1.–4.**
As noted before, the (t,i,j)-coloring of Type B is a diagonal coloring, and we can restrict our attention to the horizontal axis only. If there are no two consecutive black vertices or there are no two consecutive white vertices, then it is easy to check that this coloring is one of the following: 2-, 3-, t- or $(t+1)$-periodic. If there are two white consecutive vertices and two black consecutive vertices then, using Remark 17 and the observation following the remark, we can conclude that the only possible combinations of cases for two consecutive vertices of different colors, for two consecutive white vertices and for two consecutive black vertices respectively are the following: $A\wedge 2\wedge 3$, $B\wedge 2\wedge 1$, $B\wedge 4\wedge 3$, $C\wedge 4\wedge 1$, $D\wedge 2\wedge 1$ and $D\wedge 4\wedge 3$. In case $B\wedge 1\wedge 2$ any white point has a black point on the distance $t+1$ and any black point has a white point at distance $t+1$. So the coloring is $(t+1)$-anti-periodic. Similarly we observe that case $B\wedge 3\wedge 4$ gives a t-periodic coloring, case $D\wedge 1\wedge 2$ gives a t-anti-periodic coloring and case $D\wedge 3\wedge 4$ gives a $(t+1)$-periodic coloring. Cases $A\wedge 2\wedge 3$ and $C\wedge 1\wedge 4$ lead to contradiction.

This proves that any coloring of Type B is 2-, 3-, t- or $(t+1)$-periodic or anti-periodic. Lemmas 13, 14, 15 and 16 conclude the proof of the fact that if the coloring is of type B then it is one of the diagonal **Colorings 1.– 4.**

($\Longleftarrow$) Now, we shall show that if the diagonal coloring is one of the **Colorings 1.– 4.**, then it is an isotropic coloring of type B.
For this, it is enough to prove that, for all vertices $\mathbf{u}$ of the same color, $k(\mathbf{u})$ and $l(\mathbf{u})$ are constants. Since the number of black points in the t-neighborhood of any vertex $\mathbf{u}$ is $tk(\mathbf{u})+(t+1)l(\mathbf{u})$, the vertices of the same color would have the same number of black vertices in their t-neighborhoods.

We consider $\mathbf{u}$, its neighbor $\mathbf{u}'$ in L, and show that $l(\mathbf{u}) = l(\mathbf{u}')$, $k(\mathbf{u}) = k(\mathbf{u}')$ if $\mathbf{u}$ and $\mathbf{u}'$ are of the same color and that $l(\mathbf{u}) = l(\mathbf{u}') + c$, $k(\mathbf{u}) = k(\mathbf{u}') + c'$ for some constants c, c' if $\varphi(\mathbf{u}) = 0$ and $\varphi(\mathbf{u}') = 1$. Notice that if φ is a q-periodic or q-anti-periodic coloring, then all vertical and all horizontal lines are colored correspondingly, i.e., q-periodically or q-anti-periodically. Therefore it is sufficient to analyse only the vertices on a horizontal axis: $(x, 0)$ and $(x + 1, 0)$ for the cases when they have the same or different colors.

We prove this for the first case of the **Coloring 1.**, i.e., when φ is t-periodic, t is odd and $V(L)$ is the union of parallel monochromatic diagonals. All other cases can be handled by a very similar case analysis.

If $\varphi(x) = 0$ and $\varphi(x+1) = 0$, then $l(x) = l_1(x)+k_1(x)$, $k(x) = k_1(x)+l_1(x)$, $l(x+1) = l_1(x) + k_1(x)$ and $k(x + 1) = l_1(x) + k_1(x)$. Thus $l(x) = l(x + 1)$ and $k(x) = k(x + 1)$.

If $\varphi(x) = 1$ and $\varphi(x + 1) = 1$, then $l(x) = l_1(x) + k_1(x)$, $k(x) = k_1(x) + l_1(x) - 1$, $l(x+1) = l_1(x)+k_1(x)$ and $k(x+1) = l_1(x)+k_1(x)-1$. So, $l(x) = l(x+1)$ and $k(x) = k(x+1)$.

If $\varphi(x) = 0$ and $\varphi(x + 1) = 1$, then $l(x) = l_1(x) + k_1(x)$, $k(x) = k_1(x) + l_1(x)$, $l(x + 1) = l_1(x) + k_1(x) + 1$ and $k(x + 1) = l_1(x) + k_1(x)$. Thus $l(x) + 1 = l(x + 1)$ and $k(x) = k(x + 1)$. Thus k and l are constrants for the vertices of the fixed color. Moreover, $|i - j| = |(t + 1)l(x) + tk(x) - (t + 1)l(x + 1) - tk(x + 1)| \leq t + 1 + 1$.

This concludes the proof of Theorem 10. □

7. Conclusions

It would be interesting to describe all (t, i, j)-colorings. We conjecture that for (t, i, j)-isotropic colorings $|i - j|$ might have only values $1,\ 2,\ t,\ t + 1,\ t + 2,\ 2t + 1$. In particular, we believe that $|i - j| \leq 2$ for (t, i, j)-isotropic colorings of Type A. It would be also very interesting to study this problem for the case of the infinite triangular grid which is widely used as a model of radio-station networks.

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